On the number of k-matchings of graphs

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Abstract

In this paper an inductive formula for the number of k-matchings in graphs is derived using this formula. We concluded the number of k-matchings in special regular graphs and complete graphs.

Keywords: k-matching, matching polynomial, regular graphs.

Introduction

Let \( G = (V, E) \) be a graph in which \( V(G) \) and \( E(G) \) are the numbers of vertices and edges respectively. A matching in graph \( G \) is by definition a spanning sub graph of \( G \) whose components are vertices and edges. A k-matching is a matching with edges only. We show the number of k-matchings in a graph \( G \) by \( P(G, K) \) and assume \( P(G, 0) = 1 \).

Based on matching in a graph \( G \) we define the matching polynomial \( \mu(G, x) \) as follow

\[
\mu(G, x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k P(G, K)x^{n-2k}
\]

In which \( n \) is the number of vertices of graph \( G \).

We note that the graphs here are finite, loop less and contain no multiple edges.

The matching polynomial can be a tool for characterization of graphs. Two isomorphic graphs have the same matching polynomials that are called co-matching graphs.

However two co-matching graphs are not necessarily isomorphic¹.

Preliminaries

Finding the number of k-matching for \( k = 0, 1, \ldots, 6 \) have been done so far. For example it is easy to see \( P(G, 1) = m \) in which \( m \) is the number of edges.

For the number of two and three matching we have [2],

\[
P(G, 2) = \binom{m}{2} - \sum_{i=1}^{n} \binom{d_i}{2}
\]

\[
P(G, 3) = \binom{m}{3} - (m - 2) \sum_{i} \binom{d_i}{2} + 2 \sum_{i,j} \binom{d_i}{3} + \sum_{i,j} (d_i - 1)(d_j - 1) - N_T
\]

In which \( N_T \) is the number of triangles in \( G \).

The number of k-matchings for \( k = 4, 5, 6 \) can be found in literatures²-¹⁰.

The number of k-matchings calculated in the mentioned works shows when \( k \) grow up the formula for the number of k-matching gets very long and complicated. So calculating this number for \( k \geq 7 \) directly is not so logical and practical. Therefore in this work we derive an inductive formula for the number of k-matchings that makes it much easier to find it.

Number of k-matchings

Theorem 3.1: let \( G \) be a simple graph of order \( n \) and \( E(G) \) be the set of its edges. Then the number of k-matchings in graph \( G \) is:

\[
P(G, k) = \frac{1}{k} \sum_{i 
eq j \in E(G)} P(G - i - j, k - 1)
\]

Proof: let \( S(G, k) \) be the set of all k-matchings in \( G \). We consider an arbitrary edge \( ij \) from \( E(G) \) then we have two cases:

Case I: \( ij \) is not the component of any k-matchings in \( S(G, k) \) therefore \( P(G - i - j, k - 1) = 0 \).

Case II: \( ij \) is not the component of at least one of the k-matchings in \( S(G, k) \) so the number of matchings in \( S(G, k) \) such that \( ij \) is one of their components is \( P(G - i - j, k - 1) \)

Now according to above cases by choosing any of k-matching in \( S(G, k) \), this k-matching is counted \( k \) times so:

\[
P(G, k) = \frac{1}{k} \sum_{i,j \in E(G)} P(G - i - j, k - 1)
\]
Corollary 3.2: if \( G \) is a simple graph then:

\[
P(G, k) = \frac{1}{k!} \sum_{i_1j_1, i_2j_2, \ldots, i_kj_k} P(G - i_1 - j_1, k - 1)
\]

In which the edges \( i_1j_1, i_2j_2, \ldots, i_kj_k \) changes in the sets of edges of \( E(G), E(G - i_1 - j_1), \ldots, E(G - i_1 - j_1 - \cdots - i_{k-1} - j_{k-1}) \) respectively.

Proof: according to theorem 3.1:

\[
P(G, k) = \frac{1}{k!} \sum_{i_1j_1, i_2j_2, \ldots, i_kj_k} P(G - i_1 - j_1, k - 1)
\]

And again using the above formula for graph \( G - i_1 - j_1 \) we have:

\[
P(G - i_1 - j_1, k - 1) = \frac{1}{k - 1} \sum_{i_2j_2 \in E(G - i_1 - j_1)} P(G - i_1 - j_1 - i_2 - j_2, k - 2)
\]

So

\[
P(G, k) = \frac{1}{k!} \sum_{i_1j_1, i_2j_2} P(G - i_1 - j_1 - i_2 - j_2, k - 2)
\]

And after \( k \) times:

\[
P(G, k) = \frac{1}{k(k - 1) \ldots (1)} \sum_{i_1j_1, i_2j_2} \sum_{i_kj_k} P(G - i_1 - j_1 - \cdots - i_k - j_k, 0)
\]

But

\[
P(G - i_1 - j_1 - \cdots - i_k - j_k, 0) = 1
\]

And the theorem is proved.

Example: let \( G \) be a connected, 3-regular graph of order 8 (Figure-1), we calculate \( P(G, 4) \)

Now if \( i_1j_1 \in E(G) \) be any of edges, \( ab, bc, cd, de, ef, fg, gh, ha, af, be, ch, dg \) then the graph \( G - i_1 - j_1 \) will be isomorphic with graph \( H \) (Figure-2):

\[
P(G, 4) = \frac{12}{4!} \sum_{i_1j_1, i_2j_2} \sum_{i_3j_3} \sum_{i_4j_4} 1
\]

In which \( i_1j_1 \in E(G), i_2j_2 \in E(H), i_3j_3 \in E(H - i_2 - j_2), i_4j_4 \in E(H - i_2 - j_2 - i_3 - j_3) \) for \( i_2j_2 \in E(H) \) we consider three following cases:

Case 1: If \( i_2j_2 \) belongs to the set of edges \( E_1 = \{uv, ef\} \) then the graph \( H - i_2 - j_2 \) is isomorphic with graph \( M \) (Figure-3):

\[
\sum_{i_1j_1 \in E_1} \sum_{i_2j_2} \sum_{i_3j_3} \sum_{i_4j_4} 1 = 2 \sum_{i_3j_3} \sum_{i_4j_4} 1
\]

In which \( i_1j_1 \in E(M), i_2j_2 \in E(M - i_3 - j_3) \).

Now because \( i_1j_1 \in E(M) \) therefor \( M - i_3 - j_3 \) will be isomorphic with single edged graph (Figure-4)

\[
\sum_{i_1j_1 \in E(M)} \sum_{i_2j_2} \sum_{i_3j_3} \sum_{i_4j_4} 1 = 4 \sum_{i_4j_4} 1 (i_4j_4 = kl) = 4
\]

Therefore

\[
\sum_{i_1j_1 \in E(M)} \sum_{i_2j_2} \sum_{i_3j_3} \sum_{i_4j_4} 1 = 2 \sum_{i_3j_3} \sum_{i_4j_4} 1 = 2 \times 4 = 8
\]

Case 1: If \( i_2j_2 \) belongs to the set of edges \( E_2 = \{uw, vx, wy, xz\} \) then graph, \( H - i_2 - j_2 \) isomorphic with \( N \) (Figure-5)
Consequently in this case we have:

\[ \sum_{i_j \neq E_2} \sum_{i_j} 1 = 4 \sum_{i_j} 1 \]

in which \( i_3 \) \( j \) \( E(N) \) and \( i_4 \) \( j \) \( E(N - i_3 - j_3) \).

If \( i_3 j \) belongs to set of edges \( E_2 = \{i_j, ot\} \) then \( N - i_3 - j_3 \) is isomorphic with following single edged graph:

\[ \text{Figure-5} \]

\[ j \bullet \quad \bullet \quad t \]

So

\[ \sum_{i_j \neq E_2} \sum_{i_j} 1 = 2 \sum_{i_j} 1 (i_j = j) = 2 \]

But if \( i_3 j \) = \( j t \) then \( N - i_3 - j_3 \) will be isomorphic with the following null graph:

\[ \text{Figure-5} \]

\[ i \bullet \quad \bullet \quad o \]

And so there is no choice for \( i_j \). Therefore

\[ \sum_{i_j = j t} \sum_{i_j} 1 = 0 \]

Consequently in this case we have:

\[ \sum_{i_j \neq E_2} \sum_{i_j} 1 = 4 \sum_{i_j} 1 \]

\[ = 4 \left( \sum_{i_j \neq E_2} \sum_{i_j} 1 + \sum_{i_j = j t} \sum_{i_j} 1 \right) \]

\[ = 4 (2 + 0) = 8 \]

Case III: If \( i_j j_2 = w x \) then the graph \( H - i_2 - j_2 \) is isomorphic with graph \( R \) (Figure-6)

\[ \text{Figure-6} \]

\[ u \bullet \quad \bullet \quad v \quad \bullet \quad y \]

So

\[ \sum_{i_j = w x} \sum_{i_j} 1 = \sum_{i_j} 1 \]

In which \( i_3 j \) \( E(R), i_4 j \) \( E(R - i_3 - j_3) \)

Now since \( i_3 j \) \( E(R) \) therefor graph \( R - i_3 - j_3 \) is isomorphic with following single edged graph (Figure-7)

\[ \text{Figure-7} \]

\[ n \bullet \quad m \]

So

\[ \sum_{i_j = E(R)} \sum_{i_j} 1 = 2 \sum_{i_j} 1 (i_k j = m n) = 2 \]

Therefore

\[ \sum_{i_j = w x} \sum_{i_j} 1 = 2 \sum_{i_j} \sum_{i_j} 1 = 2 \]

Finally:

\[ P(G, 4) = \frac{12}{4!} \sum_{i_j} \sum_{i_j} 1 \]

\[ = \frac{12}{4} \left( \sum_{i_j \neq E_1} \sum_{i_j} 1 + \sum_{i_j \neq E_3} \sum_{i_j} 1 + \sum_{i_j = E(R)} \sum_{i_j} 1 \right) \]

\[ = \frac{12}{4} (8 + 8 + 2) = 9 \]

Corollary 3.3: if \( G \) be the \( 2^p \) regular graph of order \( 2^{p+1} \) then if \( k \leq 2^p + 1 \) :

\[ P(G, K) = \frac{1}{K!} \sum_{S=1}^{k} (2^p - S + 1)^2 \]

Proof: let \( m(G) \) be the number of edges. Because \( G \) is a \( 2^p \) regular graph of order \( 2^{p+1} \) so \( m(G) = 2^{2p} \)

We assume \( G_1 = G \) and choose the edge \( i_j j_1 \) from \( G_1 \) the graph

\[ G_2 = G_1 - i_1 - j_1 \]

will be of order \( 2^{p+1} - 2 \). Since the vertices \( i_1 \) and \( j_1 \) except each other are connected to \( 2^p - 1 \) other vertices so if we omit the the vertices \( i_1, j_1 \) from graph \( G \), then the \( 2^p - 1 \) \( 2^p - 1 \) \( 2^{p+1} - 2 \) vertices of graph \( G_2 \) are all of degree \( 2^p - 1 \). This means the graph \( G_2 \) is a \( 2^p - 1 \) regular graph of order \( 2^{p+1} - 2 \).therefore

\[ m(G_2) = \frac{1}{2} (2^{p+1} - 2)(2^p - 1) = (2^p - 1)^2 \]

Preceding this approach and using the same method. If we consider the edge \( i_j j_2 \) from \( (2^p - 1)^2 \) edges of graph \( G_2 \), the graph \( G_3 = G_2 - i_2 - j_2 \) is \( 2^p - 1 \) regular and of order \( 2^{p+1} - 4 \) and therefore:

\[ m(G_3) = (2^p - 2)^2 \]

After \( k \) steps, with induction we deduce that the graph \( G_k = G_{k-1} - i_{k-1} - j_{k-1} \) is \( 2^p - k + 1 \) regular of order \( 2^{p+1} - 2k + \)
2 and so \( m(G_k) = (2^p - k + 1)^2 \) but \( 2^p - k + 1 \geq 0 \) that means \( k \leq 2^p + 1 \).

Now using the corollary 2.3 we have:

\[
P(G, k) = \frac{1}{k!} \sum_{i_1,j_1 \in E(G_1)} \sum_{i_2,j_2 \in E(G_2)} \ldots \sum_{i_k,j_k \in E(G_k)} 1
\]

\[
= \frac{1}{k!} m(G_1)m(G_2) \ldots m(G_k)
\]

\[
= \frac{1}{k!} \prod_{s=1}^{k} m(G_s)
\]

\[
= \frac{1}{k!} \prod_{s=1}^{k} (2^p - s + 1)^2
\]

Corollary 3.4: if \( G \) is a complete graph of order \( n \) then with assumption \( k \leq \frac{n+1}{2} \):

\[
P(G, k) = \frac{1}{k!} \prod_{s=1}^{k} (2^p - s + 1)^2
\]

Proof: if \( G \) is a complete graph of order \( n \) then the degree of any vertex of \( G \) is \( n - 1 \) and it’s size is \( \binom{n}{2} \). Assuming \( G_1 = G \) and choosing the edge \( i_1j_1 \) from \( G_1 \) the graph \( G_2 = G_1 - i_1 - j_1 \) is a complete graph of order \( n - 2 \) and so it’s size is \( \binom{n-2}{2} \). Therefore by induction we conclude that the graph \( G_k = G_{k-1} - i_{k-1} - j_{k-1} \) is a graph of order \( n - 2k + 2 \) and size \( \binom{n-2k+2}{2} \).

But because have the degree of the vertices of \( G_k \) is \( n - 2k + 2 \) so \( n - 2k + 2 \geq 0 \) or equivalently \( \leq \frac{n+1}{2} \).

Now according to corollary 3.2

\[
P(G, k) = \frac{1}{k!} \sum_{i_1,j_1 \in E(G_1)} \sum_{i_2,j_2 \in E(G_2)} \ldots \sum_{i_k,j_k \in E(G_k)} 1
\]

\[
= \frac{1}{k!} m(G_1)m(G_2) \ldots m(G_k)
\]

\[
= \frac{1}{k!} \binom{n}{2} \binom{n-2}{2} \binom{n-2k+2}{2}
\]

\[
= \frac{n!}{2k.k!(n-2k)!}
\]

Conclusion

The result of this paper shows that a recursive formula for finding the number of matching in a graph is more applicable than a direct computation as we see in our previous work the formulas for the number of six and seven matchings are really long and complicated.

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References