Application of Homotopy Perturbation Method to Vector Host Epidemic Model with Non-Linear Incidences

Muhammad Altaf Khan1, Islam S.2, Murad Ullah2, Sher Afzal Khan3, G. Zaman1, Muhammad Arif1 and Syed Farasat Sadiq2

1Department of Mathematics, Abdul Wali Khan University Mardan, PAKISTAN
2Department of Mathematics, Islamia College University, Peshawar, PAKISTAN
3Department of Computer Science, Abdul Wali Khan University Mardan, PAKISTAN
4Department of Mathematics, University of Malakand Chakdara, Dir (lower), PAKISTAN

Available online at: www.isca.in
Received 23rd February 2013, revised 25th March 2013, accepted 21st April 2013

Abstract

In this paper, we consider an epidemic model of vector host with non-linear incidences. The spread of the disease is due to the vector such as malaria, dengue and yellow fever. We use the homotopy perturbation method to the consider model to obtain their analytical and numerical solution. Due to the importance of homotopy method a just few perturbation terms are sufficient for a reasonably accurate solution. The numerical results are presented for justification.

Keywords: Vector-host, mathematical model, Homotopy Perturbation method, numerical simulations.

Introduction

For understanding the epidemiology of infectious disease mathematical models played an important role for the infectious diseases1,2,4. This provides us the qualitative descriptions of the complicated, non-linear process of disease transmission, and also help us to obtain the dynamics of the disease and enable us to make the decision of Public health policy. World Health Statistics (2008)3 show, that some vector-borne infectious diseases as malaria, dengue and yellow fever, continue to threaten throughout the public health of many people. Most of the biological problems are inherently nonlinear. The Scientist is in search to find such Numerical methods or Perturbation method to find the exact approximate solution to these non-linear problems. Most of the nonlinear problems have not solved exactly to find their exact solution, only a few numbers of non-linear problems have the exact solution. So these non-linear problems can be solved by Numerical or traditional methods. To use analytical perturbation method, a small parameter is inserted in the equation and making use of small parameter and exerting it into the equation are the difficulties of this method. Therefore, many different powerful mathematical methods have been recently introduced to vanish the small parameters, such as artificial parameter method4,5, Zaman6 considered the model, to study the approximate solution with HPM and compare with other standard methods.

The Homotopy Analysis Method (HAM) is the well known method used for to solve the non linear equations. In the last decade, the idea of homotopy was combined with perturbation. The fundamental work was done by Liao and He. He introduced Homotopy Perturbation Method (HPM) and its application are in several problems in detail the reader are referred to4,8,9 while Ali et al.10. Presented the solution of multi point boundary values by using Optimal Homotopy Analysis Method (OHAM). These methods are free of the assumption of a such a small parameter.

In this paper, we consider the model presented in11, by applying the Homotopy perturbation method, to find the approximate solution. First, we formulate our problem and then apply the HPM to find the analytical as well as numerical solutions.

The paper is organized as follows. Section 2 is devoted to the mathematical formulation of the model and basic idea of HPM. In Section 3 the model is solved by HPM. In Section 4 we present the solution of the model numerically with a discussion and conclusion. Finally, the references are presented.

Basic idea of Homotopy perturbation method (HPM) and model framework: In this section, we explain the Homotopy perturbation method in detail and then we apply the technique of HPM to our proposed epidemic model. HPM was first time introduced by the He7,8 for solving the nonlinear differential equations.

\[ B(m) = f(d), \quad d \in \Lambda \]  

(1)

boundary conditions is

\[ \psi \left( m, \frac{\partial m}{\partial n} \right) = 0, \quad d \in \Omega \]  

(2)

Here \( B \) represents the general differential operator, \( \psi \) is the boundary operator, the analytic function is \( f(d) \), \( \Lambda \) is the boundary of the domain \( \Lambda \), \( \frac{\partial}{\partial n} \) and represent the differentiation along the normal vector drawn outward from \( \Lambda \). The operator \( B \) is divided in two parts, \( H \) is linear and \( K \) is nonlinear. So we get Equation namely (3) in the following form:

\[ H(m) + K(m) = f(d) \]  

(3)
Define the homotopy $p : \Lambda \times [0,1] \rightarrow \mathbb{R}$, that satisfies
\[ F(v, p) = (1 - p)(H(v) - H(m_0)) + pB(v) = 0, \quad (4) \]

In simplified form we write it as:
\[ F(v, p) = H(v) + p[H(m_0) + p[K(m) - f(d)]] = 0, \quad (5) \]

Here $m_0$ shows the initial approximation of $(5)$ and $p$ is the embedding parameter, $p \in [0,1]$. We see that
\[ F(v, 0) = [H(v) - H(m_0)] = 0, \quad F(v, 1) = [B(v) - f(d)] = 0 \]

for putting $p=0$, we obtain, $F(v, 0) = [H(v) - H(m_0)]$ and using $p=1$, we get $F(v, 1) = [B(v) - f(d)]$.

Applying the perturbation technique, consider $p$ is the smallest parameter then the solution of equation $(4)$ can be considered as series in $p$, which is given by
\[ v = v_o + pv_1 + p^2v_2 + p^3v_3 ..., \quad (7) \]

When $p$ approaches 1 the equation $(4)$ becomes the original equation $(3)$ and $(7)$ becomes the approximate solution of $(3)$ given by
\[ m = \lim_{p \to 1} v = v_o + pv_1 + p^2v_2 + p^3v_3 ..., \quad (8) \]

The series $(8)$ is convergent for most of the cases, for reader see $^{7,8}$.

Now we formulate our problem here, assume that $S(t)$ the number of susceptible human at time $t$; $I(t)$ is the number of infected human at time $t$ and $R_0(t)$ represents the recovered individuals. The total population size is denoted by $N$, with $N_S = S(t) + I(t) + R_0(t)$. For vector population, let $S_v(t)$ are susceptible vector and $I_v(t)$ are infectious vector at time $t$ and $d = \gamma_S + \gamma_I$. The total population size of vector population is denoted by $N_v$ with $N_v = S_v(t) + I_v(t)$.

The governing differential equation is given by
\[ \begin{align*}
S_h' &= \mu K - \beta_h S_h I_h - \mu S_h, \\
I_h' &= \frac{\beta_h S_h I_h}{1 + \alpha_I} - (\mu + d) I_h, \\
R_h' &= \Gamma I_h - \mu R_h, \\
S_v' &= \Lambda - \frac{\beta_v S_v I_v}{1 + \alpha_I} - n S_v, \\
I_v' &= \frac{\beta_v S_v I_v}{1 + \alpha_I} - n I_v.
\end{align*} \tag{9} \]

With the initial conditions
\[ S_h(0) \geq 0, I_h(0) \geq 0, R_h(0) \geq 0, S_v(0) \geq 0, I_v(0) \geq 0 \tag{10} \]

The parameters used in the model $(9)$, $\mu K$ is the recruitment rate of human population, $\beta_h$ represents the transmission rate from vector to human, $\beta_v$ is the transmission from human to vector, the natural death rate for the human is denoted by $\mu$, $\gamma_S$ represents the treatment rate for human, the rate of recovery from infection is shown by $\gamma_I$. The natural mortality rate for vector is $n$, the level at which the force of infection saturated is denoted by $\alpha_I$ and $\alpha_2$ and the birth rate for the vector is $\Lambda$.

**Solution of Model by HPM**

In this section, we use the model $(9)$ by applying the homotopy perturbation method. For this first we choose $S_h(t) = S, I_h(t) = I, R_h(t) = R, S_v(t) = Q$ and $I_v(t) = W$ and then apply the method, which is given by:
\[ \begin{align*}
LS(t) - LS^0(t) &= p \left( \mu K - \frac{\beta_h SW}{1 + \alpha_I W} - \mu S - LS^0(t) \right), \\
L(t) - LI^0(t) &= p \left( \frac{\beta_h SW}{1 + \alpha_I W} - (\mu + d) I - LI^0(t) \right), \\
LR(t) - LR^0(t) &= p dI - \mu R - LR^0(t), \\
LQ(t) - LQ^0(t) &= p (\Lambda - \frac{\beta_v Q I}{1 + \alpha_I I} - n Q - LQ^0(t)), \\
LW(t) - LW^0(t) &= p \left( \frac{\beta_v Q I}{1 + \alpha_I I} - n W - LW^0(t) \right)
\end{align*} \tag{11} \]

Here we define the operator $L = \frac{d}{dt}$. The initial data we consider by
\[ S(t) = S(0), I(t) = I(0), R(t) = R(0), Q(t) = Q(0), \text{ and } W(t) = W(0) \tag{12} \]

Assuming the solution of $(11)$ in the form
\[ \begin{align*}
S(t) &= S^*(t) + pS_1^*(t) + p^2S_2^*(t) + ..., \\
I(t) &= I^*(t) + pI_1^*(t) + p^2I_2^*(t) + ..., \\
R(t) &= R^*(t) + pR_1^*(t) + p^2R_2^*(t) + ..., \\
Q(t) &= Q^*(t) + pQ_1^*(t) + p^2Q_2^*(t) + ..., \\
W(t) &= W^*(t) + pW_1^*(t) + p^2W_2^*(t) + ...
\end{align*} \tag{13} \]

Making use of $(13)$ in $(11)$ and comparing the coefficient of the same power, we get
\[ \begin{align*}
LS^*(t) - LS^0(t) &= 0, \\
LI^*(t) - LI^0(t) &= 0, \\
LR(t) - LR^0(t) &= 0, \\
LQ(t) - LQ^0(t) &= 0, \\
LW(t) - LW^0(t) &= 0.
\end{align*} \tag{14} \]

And
\[ \begin{align*}
LS_1^*(t) &= (\mu K - \frac{\beta_h S^0(t) I^0(t)}{1 + \alpha_I I^0(t)} - \mu S^0(t) - LS^0(t)), \\
LI_1^*(t) &= \left( \frac{\beta_h S_1^*(t) I_1^*(t)}{1 + \alpha_I I_1^*(t)} - (\mu + d) I_1^*(t) - LI^0(t) \right), \\
LR_1^*(t) &= \left( dI^*_1(t) - \mu R^*_1(t) - LR^0(t) \right), \\
LQ_1^*(t) &= \left( \frac{\beta_v Q_1^*(t) I^*_1(t)}{1 + \alpha_I I^*_1(t)} - nQ^*_1(t) - LQ^0(t) \right), \\
LW_1^*(t) &= \left( \frac{\beta_v Q^*_1(t) I^*_1(t)}{1 + \alpha_I I^*_1(t)} - nW^*_1(t) - LW^0(t) \right)
\end{align*} \tag{15} \]

With the conditions
\[ S_1^*(t) = 0, I_1^*(t) = 0, R_1^*(t) = 0, Q_1^*(t) = 0, \text{ and } W_1^*(t) = 0. \tag{16} \]
And
\[
L S_2^2(t) = \left( -\frac{\beta_h [S_2^2(t)Q_1^2(t) + S_1^2(t)Q_2^2(t)]}{\alpha_1 Q_1^2(t)} - \mu S_1^2(t), \right)
\]
\[
L S_2^2(t) = \left( -\frac{\beta_h [S_2^2(t)Q_1^2(t) + S_1^2(t)Q_2^2(t)]}{\alpha_2 Q_1^2(t)} - (\mu + d)I_1^2(t), \right)
\]
\[
L R_2^2(t) = \left( -\frac{\beta_h [I_2^2(t)Q_1^2(t) + I_1^2(t)Q_2^2(t)]}{1 + \alpha_2 I_1^2(t)} - nQ_1^2(t), \right)
\]
\[
L W_2^2(t) = \left( -\frac{\beta_h [I_2^2(t)Q_1^2(t) + I_1^2(t)Q_2^2(t)]}{1 + \alpha_2 I_1^2(t)} - nW_1^2(t), \right)
\]
with the conditions
\[
S_2^2(t) = 0, I_2^2(t) = 0, R_2^2(t) = 0, Q_2^2(t) = 0, and W_2^2(t) = 0.
\]

In similar fashion, we obtain
\[
L S_1^2(t) = \left( -\frac{\beta_h [S_1^2(t)Q_1^2(t) + S_1^2(t)Q_1^2(t) + S_2^2(t)Q_2^2(t)]}{\alpha_1 Q_2^2(t)} - \mu S_2^2(t), \right)
\]
\[
L I_1^2(t) = \left( \frac{\beta_h [S_1^2(t)Q_1^2(t) + S_1^2(t)Q_1^2(t) + S_2^2(t)Q_2^2(t)]}{\alpha_2 I_1^2(t)} - (\mu + d)I_1^2(t), \right)
\]
\[
L R_1^2(t) = \left( -\frac{\beta_h [I_1^2(t)Q_1^2(t) + I_1^2(t)Q_1^2(t) + I_2^2(t)Q_2^2(t)]}{1 + \alpha_2 I_1^2(t)} - nQ_1^2(t), \right)
\]
\[
L W_1^2(t) = \left( -\frac{\beta_h [I_1^2(t)Q_1^2(t) + I_1^2(t)Q_1^2(t) + I_2^2(t)Q_2^2(t)]}{1 + \alpha_2 I_1^2(t)} - nW_1^2(t), \right)
\]
To find the solution, we put p=1 in the system (13), we get
\[
S^*(t) = S_1^2(t) + S_2^2(t) + S_3^2(t) + \ldots,
\]
\[
I^*(t) = I_1^2(t) + I_1^2(t) + I_2^2(t) + \ldots,
\]
\[
R^*(t) = R_1^2(t) + R_2^2(t) + R_3^2(t) + \ldots,
\]
\[
Q^*(t) = Q_1^2(t) + Q_2^2(t) + Q_3^2(t) + \ldots,
\]
\[
W^*(t) = W_1^2(t) + W_1^2(t) + W_2^2(t) + \ldots.
\]

The convergence of HPM is rapid, for a few iterations of both linear and non-linear.

**Zeroth order solution or P^0**
\[
S_2^2(t) = 130, \quad I_2^2(t) = 80, \quad R_2^2(t) = 100, \quad Q_2^2(t) = 220, \quad and \quad W_2^2(t) = 200.
\]

**First order solution or P^1**
\[
S_1^2(t) = \left( \mu K - \frac{\beta_h e_1 e_5}{1 + \alpha_1 e_5} - \mu e_1 \right) t,
\]
\[
I_1^2(t) = \left( \frac{\beta_h e_1 e_5}{1 + \alpha_1 e_5} - (\mu + d) e_2 \right) t,
\]
\[
R_1^2(t) = (de_2 - \mu e_3) t,
\]
\[
Q_1^2(t) = \left( \lambda - \frac{\beta_v e_4 e_5}{1 + \alpha_2 e_5} - ne_4 \right) t,
\]
\[
W_1^2(t) = \left( \frac{\beta_v e_4 e_5}{1 + \alpha_3 e_5} - ne_5 \right) t,
\]

**Numerical Results**

In this section, we solve our proposed model numerically by using the Runge-Kutta 4th order scheme with the positive initial conditions. In the numerical simulation the values assigned to the parameter are presented in table-1. The solution of the model is presented in the form of plots. In our simulations the figure-1 represent the susceptible human individuals in the model. Figure-2 represents the infected human individuals. Figure-3 shows the population of recovered human. The population of susceptible vector is shown in figure-4 and the population of infected vector is represented by figure-5.
\[ W_2(t) = \beta_0 \left( \frac{\left( \Lambda - \frac{\beta_h e_2 e_4}{1 + \alpha_2 e_2} - ne_4 \right) e_2 + \left( \frac{\beta_h e_2 e_5}{1 + \alpha_1 e_5} - (\mu + d)e_2 \right) e_4}{\alpha_2 \left( \frac{\beta_h e_1 e_5}{1 + \alpha_1 e_5} - (\mu + d)e_2 \right)} \right) \frac{t^2}{2} - n \left( \frac{\beta_h e_2 e_4}{1 + \alpha_2 e_2} - ne_4 \right) \frac{t^2}{2} \]

Table 1
Parameter and their values used in the numerical solution

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu K )</td>
<td>Represent the intrinsic growth rate of the human population</td>
</tr>
<tr>
<td>( \beta_h )</td>
<td>Transmission rate from vector to human</td>
</tr>
<tr>
<td>( \beta_v )</td>
<td>Transmission from human to vector</td>
</tr>
<tr>
<td>( \mu )</td>
<td>A natural deathrate of a human</td>
</tr>
<tr>
<td>( \gamma_1 )</td>
<td>Treated rate of human</td>
</tr>
<tr>
<td>( \gamma_0 )</td>
<td>A recovery rate of infection of human</td>
</tr>
<tr>
<td>( n )</td>
<td>The mortality rate of vector</td>
</tr>
<tr>
<td>( \alpha_1, \alpha_2 )</td>
<td>The level at which the force of infection saturates</td>
</tr>
<tr>
<td>( \Lambda )</td>
<td>Recruitment rate of vector</td>
</tr>
</tbody>
</table>

Figure 1
Shows the susceptible individuals

Figure 2
Represent the Infected human individuals in the population.
Figure-3
Represent the population of recovered individuals

Figure-4
Represent the population of susceptible individuals

Figure-5
Represent the population of infected individuals
Conclusion

In this paper, we have presented an epidemic model and applied the homotopy perturbation techniques. First, we have explained the techniques in detail and then we applied to our proposed model. The importance of homotopy perturbation method is that for a system of nonlinear differential equation just a few iterations is enough for a best and reliable results. We have formulated the problem, and then by comparing the coefficient, the solution to the zeroth, first and second order was obtained. The obtained zeroth order solution, was used to obtain the first order solution. Similarly the second order solution was obtained by using the first order. Then we solved the model numerically and the results are presented in the form of plots for justification purpose.

References