Short Communication

Some New Results on $T^1$, $T^2$ and $T^4$-AG-groupoids

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Abstract

In this article we investigate some basic properties of newly discovered classes of AG-groupoid. We consider three classes that include $T^1$, $T^2$ and $T^4$-AG-groupoids. We prove that every $T^4$-AG-groupoid is Bol*-AG-groupoid. We further investigate that $T^1$ and $T^4$-AG-groupoids are paramedial and hence are left nuclear square AG-groupoids. We also prove that $T^1$ and $T^4$ are transitively commutative AG-groupoids and $T^1$-AG-3-band is a semigroup.

Keywords: AG-groupoid, LA-semigroup, AG-group, types of AG-groupoid, nuclear square, $T^1$, $T^2$ and $T^4$-AG-groupoids.

Introduction

A groupoid $S$ is called an AG-groupoid if it satisfies the left invertive law\(^1\): $(ab)c = (cb)a$. This structure is also known as left almost semi-group\(^2\) (LA-semigroup), left invertive groupoid\(^3\) and right modular groupoid\(^4\). An AG-groupoid $S$ is called AG-3-band if $a(aa) = (aa)a = a \ \forall \ a \in S$. In this paper we are going to investigate some interesting properties of newly discovered classes of namely: $T^1$, $T^2$ and $T^4$-AG-groupoids\(^5\). An AG-groupoid $S$ always satisfies the medial law\(^6\): $(ab)(cd) = (ac)(bd)$, while an AG-groupoid $S$ with left identity $e$ always satisfies paramedial law: $(ab)(cd) = (db)(ca)$. An AG-groupoid $S$ is called transitive-commutative\(^7\) if:

\[
\forall \ a, b, c \in S, \ ab = ba, \ bc = cb \ implies \ ac = ca.
\]

Recently some new classes of AG-groupoid have been discovered\(^8\) that are: Bol*, $T^1$, $T^2$, $T^3$ and $T^4$-AG-groupoid and some others. Here we consider $T^1$, $T^2$ and $T^4$-AG-groupoids to further investigate them. An AG-groupoid $S$ is called $T^1$-AG-groupoid if $\forall \ a, b, c, d \in S, \ ab = cd$ implies $ba = dc$ and is called $T^2$-AG-groupoid if $\forall \ a, b, c, d \in S, \ ab = cd$ implies $ac = bd$. An AG-groupoid $S$ is called forward $T^4$-AG-groupoid denoted by $T^4_f$ if $\forall \ a, b, c, d \in S, \ ab = cd$ implies $ad = cb$, and is called backward $T^4$-AG-groupoid, denoted by $T^4_b$, if $\forall \ a, b, c, d \in S, \ ab = cd$ implies $da = bc$. An AG-groupoid $S$ is called $T^4$-AG-groupoid if it is both forward and backward $T^4$-AG-groupoid. An AG-groupoid $S$ is called left nuclear square\(^9\) if $\forall \ a, b, c \in S, \ a^2(bc) = (a^2b)c$. Right nuclear and nuclear square can be defined analogously. A groupoid $S$ is called Bol*-groupoid if it satisfies the identity $a(bc) d = (ab) c d$. A groupoid $S$ is called left cancellative\(^10\) if $\forall \ a, b, c \in S, \ ab = ac$ implies $b = c$. Right cancellative and cancellative AG-groupoid can be defined similarly.

Properties of $T^1$, $T^2$ and $T^4$-AG-groupoids

It is proved that every $T^1$-AG-groupoid is Bol*-AG-groupoid, and that every Bol*-AG-groupoid is paramedial AG-groupoid\(^6\). Here we proceed to prove that every $T^1$-AG-groupoid is paramedial but the converse is not true.

**Theorem 1**: Every $T^1$- AG-groupoid is paramedial-AG-groupoid

**Proof**. Let $S$ be a $T^1$-AG-groupoid, and let $a, b, c, d \in S$.

Then by definition of $T^1$-AG-groupoid

$ab = cd \Rightarrow ba = dc$.

Now since,

$(ab) cd = ac bd \quad \text{(by medial law)}$

$\Rightarrow cd.ab = bd.ac \quad \text{(S is T^1 - AG - groupoid)}$

$= ba.dc \quad \text{(by medial law)}$

$\Rightarrow ab.cd dc.ba \quad \text{(S is T^1 - AG - groupoid)}$

$= db.ca \quad \text{(by medial law)}$

$\Rightarrow ab.cd = db.ca$.

Hence $S$ is paramedial-AG-groupoid.

Here is an example of paramedial AG-groupoid that is not $T^1$- AG-groupoid.

**Example 1**. Paramedial AG-groupoid of order 3 which is not $T^1$-AG-groupoid.
Consider \( ac = bd \) \( \Rightarrow ab = cd \) \( (\text{by } T^2 - \text{AG-groupoid}) \)
\( \Rightarrow ba = cd \) \( (\text{as } ab = ba) \)
\( \Rightarrow bc = ad \) \( (\text{by } T^2 - \text{AG-groupoid}) \)
\( \Rightarrow cb = ad \) \( (\text{as } cb = bc) \)
\( \Rightarrow ca = bd \) \( (\text{by } T^2 - \text{AG-groupoid}) \)
\( \Rightarrow ac = ca \) \( (\text{by Equation (1) and (2)}) \)
Hence \( S \) is transitively commutative AG-groupoid.

**Theorem 5:** Let \( S \) be an AG-groupoid with left identity \( e \) such that \( a^2 = e \; \forall \; a \in S \). Then \( S \) is \( T^2 \)-AG-groupoid.

**Proof.** Let \( S \) be an AG-monoid with left identity \( e \) such that \( a^2 = e \). Let \( a, b, c, d \in S \) and \( ab = cd \).

Then, \( ac = (ea)c = (ca)e = (ca) \) \( (\text{by left invertive law}) \)
\( = ca.bd \) \( (\text{by assumption}) \)
\( = cb.ab \) \( (\text{by medial law}) \)
\( = cb.cd \) \( (\text{by Equation (3)}) \)
\( = cc.bd \) \( (\text{by medial law}) \)
\( = e.bd \) \( (\text{by assumption}) \)
\( \Rightarrow ac = bd \). \( (e \text{ is left identity}) \)
Hence \( S \) is \( T^2 \)-AG-groupoid.

We will use the following lemmas to prove some further properties of \( T^2 \)-AG-groupoid.

**Lemma 1:** Every \( T^1 \)-AG-groupoid is \( Bol^* \)-AG-groupoid.

**Lemma 2:** Every \( T^2 \)-AG-groupoid is \( T^1 \)-AG-groupoid.

Since, by Lemma 1 and 2, we know that every \( T^1 \)-AG-groupoid is \( Bol^* \)-AG-groupoid and every \( T^2 \)-AG-groupoid is \( T^1 \)-AG-groupoid. We immediately have the following result.

**Corollary 3:** Every \( T^2 \)-AG-groupoid is \( Bol^* \)-AG-groupoid.

**Proof:** Let \( S \) be a \( T^2 \)-AG-groupoid. Then \( \forall \ a, b, c, d \in S \) we have \( ab = cd \) \( \Rightarrow ac = bd \).

Now consider,
\( (ab)c = (ab)c \) \( (\text{by left invertive law}) \)
\( \Rightarrow d(ab)c = d(ab)c \) \( (\text{by left invertive law}) \)
\( = d(ab)c = d(ab)c \) \( (\text{by assumption}) \)
\( = d(ab)c = d(ab)c \) \( (\text{by medial law}) \)
\( \Rightarrow (ab)c = (ab)c \) \( (\text{by left invertive law}) \)
\( = (ab)c = (ab)c \) \( (\text{by assumption}) \)
\( = (ab)c = (ab)c \) \( (\text{by left invertive law}) \)
\( \Rightarrow (ab)c = (ab)c \) \( (\text{by assumption}) \)
Hence \( S \) is a semigroup.

**Theorem 4:** Every \( T^2 \)-AG-groupoid is transitively commutative AG-groupoid.

**Proof:** Let \( S \) be \( T^2 \)-AG-groupoid. Then \( \forall \ a, b, c, d \in S \), we have
\( ab = cd \) \( \Rightarrow ac = bd \),
\( \Rightarrow ab = ba \), \( bc = cb \).
Corollary 4: Every $T^1$-AG-groupoid is paramedial AG-groupoid.

Corollary 5: Every $T^2$-AG-groupoid is left nuclear square AG-groupoid.

The following result gives an interesting relation between $T^4$. AG-groupoids and Bol*-AG-groupoids.

Theorem 6: Every $T^4$-AG-groupoid is Bol*-AG-groupoid.

Proof: Let $S$ be $T^4$-AG-groupoid, and let $a, b, c, d \in S$. Then by definition $T^4$-AG-groupoid
\[
ab = cd \Rightarrow ad = cb \quad (S \text{ is } T^4_1 - AG - \text{groupoid})
\]
\[
ab = cd \Rightarrow da = bc \quad (S \text{ is } T^4_2 - AG - \text{groupoid})
\]

Now let, $(ab)c = ad\cdot bc \text{ (by left invertive law)}$
\[
\Rightarrow (ab)c = ab\cdot dc \quad (S \text{ is } T^4_3 - AG - \text{groupoid})
\]
\[
\Rightarrow (ab)c = d\cdot ab \quad (S \text{ is } T^4_4 - AG - \text{groupoid})
\]
\[
\Rightarrow (ab)c = d\cdot ab \quad (S \text{ is } T^4_5 - AG - \text{groupoid})
\]

Hence $S$ is Bol*-AG-groupoid.

Since each Bol*-AG-groupoid is paramedial and thus is left nuclear square, whence using Theorem 6 we immediately have the following:

Corollary 6: Every $T^4$-AG-groupoid is paramedial AG-groupoid.

Corollary 7: Every $T^4$-AG-groupoid is left nuclear square AG-groupoid.

Next we prove that the class of transitively commutative AG-groupoids contains the class of all $T^4$-AG-groupoids.

Theorem 7: Every $T^4$-AG-groupoid is transitively commutative AG-groupoid.

Proof. Let $S$ be $T^4$-AG-groupoid. Then $\forall a, b, c, d \in S$, we have
\[
ab = cd \Rightarrow ad = cb \quad (S \text{ is } T^4_1 - AG - \text{groupoid})
\]
\[
ab = cd \Rightarrow da = bc \quad (S \text{ is } T^4_2 - AG - \text{groupoid})
\]

Now let, $ab = ba \Rightarrow cb = bc$. Consider $ab = ba \Rightarrow cb = bc$. Applying definition of $T^4$-AG-groupoid, we have, $ac = ca$.

Hence $S$ is transitively commutative AG-groupoid.

It is known that every $T^1$-AG-groupoid is $AG^{**}$-groupoid. Here we prove that every cancellative $AG^{**}$-groupoid is $T^1$-AG-groupoid.

Theorem 8: Every cancellative $AG^{**}$-groupoid is $T^1$-AG-groupoid.

Proof: Let $S$ be a cancellative $AG^{**}$-groupoid and let $x$ be a cancellative element of $S$. Then $\forall a, b, c, d \in S$, let $ab = cd$.

Now since,
\[
x^2(ba) = b(x^2)a \quad (S \text{ is } AG^{**} - \text{groupoid})
\]
\[
x^2(ab) = b(ax).x \quad (by \text{ left invertive law})
\]
\[
x^2(ab) = ax.bx \quad (S \text{ is } AG^{**} - \text{groupoid})
\]
\[
x^2(ab) = ab.xx \quad (by \text{ left invertive law})
\]
\[
x^2(ab) = ab.xx \quad (S \text{ is } AG^{**} - \text{groupoid})
\]
\[
x^2(ab) = ab.xx \quad (by \text{ assumption})
\]
\[
x^2(ab) = x(cd).x \quad (S \text{ is } AG^{**} - \text{groupoid})
\]
\[
x^2(ab) = x(xd).c \quad (by \text{ left invertive law})
\]
\[
x^2(ab) = xd.xc \quad (S \text{ is } AG^{**} - \text{groupoid})
\]
\[
x^2(ab) = xx.dc \quad (by \text{ left invertive law})
\]
\[
x^2(ab) = x^2.dc \quad (S \text{ is cancellative})
\]

Hence $S$ is $T^1$-AG-groupoid.

Conclusion

Many new classes of AG-groupoids have been discovered recently. Enumeration has also been done of these new classes up to order 6. All this has attracted researchers of the field to investigate the newly discovered classes in detail. This current article investigates the ideas of $T^1$, $T^2$ and $T^4$-AG-groupoids. We investigate that every $T^4$-AG-groupoid is Bol*-AG-groupoid. We further investigate that $T^1$ and $T^4$-AG-groupoids are paramedial and hence are left nuclear square. We also prove that $T^1$ and $T^4$ are transitively commutative AG-groupoid and $T^4$-AG-3-band is a semigroup.

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