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Short Communication

Some New Results on T^1, T^2 and T^4-AG-groupoids

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Abstract

In this article we investigate some basic properties of newly discovered classes of AG-groupoid. We consider three classes that include T^1, T^2 and T^4 -AG-groupoids. We prove that every T^4-AG-groupoid is Bol-AG-groupoid. We further investigate that T^1 and T^4-AG-groupoids are paramedial and hence are left nuclear square AG-groupoids. We also prove that T^1 and T^4 are transitively commutative AG-groupoids and T^1-AG-3-band is a semigroup.*

Keywords: AG-groupoid, LA-semigroup, AG-group, types of AG-groupoid, nuclear square, T^1, T^2 and T^4 -*AG-groupoids*.

Introduction

A groupoid S is called an AG-groupoid if it satisfies the left invertive law[1]: $(ab)c = (cb)a$. This structure is also known as left almost semi-group[2] (LA-semigroup), left invertive groupoid[3] and right modular groupoid[3]. An AG-groupoid S is called AG-3-band if $a(aa) = (aa)a = a\ \forall\ a \in S$. In this paper we are going to investigate some interesting properties of newly discovered classes of namely: T^1, T^2 and T^4 AG-groupoids[4]. An AG-groupoid S always satisfies the medial law[1]: $(ab)(cd) = (ac)(bd)$, while an AG-groupoid S with left identity e always satisfies paramedial law: $(ab)(cd) = (db)(ca)$. An AG-groupoid S is called transitively-commutative[4] if:

$\forall\, a, b, c \in S,\ \ ab = ba,\ \ bc = cb\ \ \text{implies}\ \ ac = ca.$

Recently some new classes of AG-groupoid have been discovered[5, 6] that are; Bol*, T^1, T^2, T^3 and T^4- AG-groupoid and some others. Here we consider T^1, T^2 and T^4 -AG-groupoids to further investigate them. An AG-groupoid S is called T^1-*AG-groupoid* if $\forall\, a, b, c, d \in S$, $ab = cd$ *implies* $ba = dc$ and is called T^2-*AG-groupoid* if $\forall\, a, b, c, d \in S$, $ab = cd$ *implies* $ac = bd$. An AG-groupoid S is called *forward* T^4-*AG-groupoid* denoted by T_f^4 if $\forall\, a, b, c, d \in S$, $ab = cd$ *implies* $ad = cb$, and is called *backward* T^4-*AG-groupoid*, denoted by T_b^4, if $\forall\, a, b, c, d \in S$, $ab = cd$ *implies* $da = bc$. An AG-groupoid S is called T^4-*AG-groupoid* if it is both forward and backward T^4-AG-groupoid. An AG-groupoid S is called *left nuclear square*[4] if $\forall\, a, b, c \in S$, $a^2(bc) = (a^2b)c$. Right nuclear and nuclear square can be defined analogously. A groupoid S is called *Bol*-groupoid* if it satisfies the identity $a(bc.d) = (ab.c)d$. A groupoid S is called left cancellative[4] if $\forall\, a, b, c \in S$, $ab = ac$ *implies* $b = c$. Right cancellative and cancellative AG-groupoid can be defined similarly.

It should be noted that various algebraic structures can be constructed from each other by defying a suitable relation between them. A similar article[7] by the authors can be seen that how some new classes of AG-groupoids can be constructed from other known classes. A generalization of cancellative AG-groupoid has been done as quasi-cancellativity[8]. AG-groupoids; generalize commutative semigroups and have applications in flock theory[9] and some in geometry[4].

Properties of T^1, T^2 and T^4 -AG-groupoids

It is proved that every T^1-AG-groupoid is Bol*-AG-groupoid[5], and that every Bol*-AG-groupoid is paramedial AG-groupoid[6]. Here we proceed to prove that every T^1-AG-groupoid is paramedial but the converse is not true.

Theorem 1: *Every T^1- AG-groupoid is paramedial-AG-groupoid*

Proof. Let S be a T^1-AG-groupoid, and let $a, b, c, d \in S$.
Then by definition of T^1-AG-groupoid
$ab = cd \quad \Rightarrow \quad ba = dc.$
Now since,

$$
\begin{aligned}
ab.cd &= ac.bd && (by\ medial\ law)\\
\Rightarrow cd.ab &= bd.ac && (S\ is\ T^1 - AG - groupoid)\\
&= ba.dc && (by\ medial\ law)\\
\Rightarrow ab.cd\ \ & dc.ba && (S\ is\ T^1 - AG - groupoid)\\
&= db.ca && (by\ medial\ law)\\
\Rightarrow ab.cd &= db.ca.
\end{aligned}
$$

Hence S is paramedial-AG-groupoid.

Here is an example of paramedial AG-groupoid that is not $\boldsymbol{T^1}$-AG-groupoid.

Example 1. Paramedial AG-groupoid of order 3 which is not T^1-AG-groupoid.

*	1	2	3
1	1	1	1
2	1	1	1
3	2	2	2

Since each paramedial is left nuclear sqauare[4]. The following corollary is now an obvious fact.

Corollary 1: *Every T^1-AG-groupoid is left nuclear square AG-groupoid.*

Theorem 2. *Bol* -AG-groupoid with left identity is T^1- AG-groupoid.*

Proof. Let S be a Bol* -AG-groupoid with left identity e, and $a,b,c,d \in S$. Let $ab = cd$.

Then,

$ba = e(eb.a)$ $(e \text{ is left identity})$
$= (e.eb)a$ $(S \text{ is } Bol^* - AG - groupoid)$
$= (eb)a$ $(e \text{ is left identity})$
$= (ab)e$ $(by \text{ left invertive law})$
$= cd.e$ $(by \text{ assumption})$

$= ed.c$ $(by \text{ left invertive law})$
$\Rightarrow ba = dc.$
Hence S is T^1-AG-groupoid.

Corollary 2. *Bol*-AG-groupoid with left identity is T^3- AG-groupoid.*

Theorem 3. *Every T^1-AG-3-band is semigroup.*

Proof. Let S be T^1-AG-groupoid that is also AG-3-band. Then $\forall\ a,b,c,d \in S$;

$ab = cd \Rightarrow ba = dc$ Now since,
$(ab)c = (cb)a$ $(by \text{ left invertive law})$
$\Rightarrow c(ab) = a(cb)$ $(S \text{ is } T^1 - AG - groupoid)$
$= ((aa)a)(cb)$ $(S \text{ is } AG - 3 - band)$
$= (aa)c.ab$ $(by \text{ medial law})$
$\Rightarrow c(ab) = (aa)c.ab$
$\Rightarrow (ab)c = ab.(aa)c$ $(S \text{ is } T^1 - AG - groupoid)$
$= a(aa).bc$ $(by \text{ medial law})$
$(ab)c = a(bc)$ $(by\ AG - 3 - band)$
$\Rightarrow (ab)c \quad a(bc).$
Hence S is a semigroup[10].

Theorem 4: Every *T^2-AG-groupoid is transitively commutative AG-groupoid.*

Proof: Let S be T^2 -AG-groupoid. Then $\forall\ a,b,c,d \in S$, we have

$ab = cd \Rightarrow ac = bd$
Let $ab = ba, \quad bc = cb.$

Consider $ac = bd$ (1)
$\Rightarrow ab = cd$ $(by\ T^2 - AG - groupoid)$
$\Rightarrow ba = cd$ $(as\ ab = ba)$
$\Rightarrow bc = ad$ $(by\ T^2 - AG - groupoid)$
$\Rightarrow cb = ad$ $(as\ cb = bc)$
$\Rightarrow ca = bd$ $(by\ T^2 - AG - groupoid)$ (2)
$\Rightarrow ac = ca$ $(by\ Equation\ (1)\ and\ (2))$
Hence S is transitively commutative AG-groupoid.

Theorem 5: *Let S be an AG-groupoid with left identity e such that $a^2 = e\ \forall a \in S$. Then S is T^2-AG-groupoid.*

Proof. Let S be an AG-monoid with left identity e such that $a^2 = e$. Let $a,b,c,d \in S$ and $ab = cd$. (3)

Then, $ac = (ea)c$
$= (ca)e$ (by left invertive law)
$= ca.bb$ (by assumption)
$= cb.ab$ (by medial law)
$= cb.cd$ (by Equation (3))
$= cc.bd$ (by medial law)
$= e.bd$ (by assumption)
$\Rightarrow ac = bd.$ $(e \text{ is left identity})$
Hence S is T^2 -AG-groupoid.

We will use the following lemmas to prove some further properties of T^2-AG-groupoid.

Lemma 1: *Every T^1 -AG-groupoid is Bol*-AG-groupoid*[4].

Lemma 2: Every T^2-AG-groupoid is T^1-AG-groupoid[4].

Since, by Lemma 1 and 2, we know that every T^1 -AG-groupoid is Bol* -AG-groupoid and every T^2 -AG-groupoid is T^1 -AG-groupoid. We immediately have the following result.

Corollary 3: *Every T^2-AG-groupoid is Bol*-AG-groupoid.*

Proof: Let S be a T^2-AG-groupoid. Then $\forall\ a,b,c,d \in S$ we have $ab = cd \Rightarrow ac = bd.$
Now consider,
$(ab.c)d \quad dc.ab$ $(by \text{ left invertive law})$ (4)
$\Rightarrow d(ab.c) = ab.dc$ $(by\ Lemma\ 2)$
$= (dc.b)a$ $(by \text{ left invertive law})$
$\Rightarrow d(dc.b) = (ab.c)a$ $(S \text{ is } T^2 - AG - groupoid)$
$(dc.b)d = a(ab.c)$ $(by\ Lemma\ 2)$
$\Rightarrow (dc.b)a = d(ab.c)$ $(S \text{ is } T^2 - AG - groupoid)$
$\Rightarrow a(dc.b) = (ab.c)d$ $(by\ Lemma\ 2)$
$\Rightarrow a(dc.b) = dc.ab$ $(by \text{ left invertive law})$ (5)
$\Rightarrow (ab.c)d = a(dc.b)$ $(by\ Eqn\ (4)\ and\ (5))$
$\Rightarrow (ab.c)d = a(bc.d)$ $(by \text{ left invertive law})$
Hence S is Bol*-AG-groupoid.

Since every T^2-AG-groupoid is T^1-AG-groupoid[4] and every T^1-AG-groupoid is paramedial AG-groupoid by Theorem 1 and is left nuclear square by Corollary 1, thus we have the following:

Corollary 4: *Every T^2-AG-groupoid is paramedial AG-groupoid.*

Corollary 5: *Every T^2-AG-groupoid is left nuclear square AG-groupoid.*

The following result gives an interesting relation between T^4-AG-groupoids and Bol* -AG-groupoids.

Theorem 6: *Every T^4-AG-groupoid is Bol* -AG-groupoid.*

Proof: Let S be T^4-AG-groupoid, and let $a, b, c, d \in S$. Then by definition T^4-AG-groupoid

$ab = cd \Rightarrow ad = cb \quad (S \text{ is } T_f^4 - AG - groupoid)$

$ab = cd \Rightarrow da = bc \quad (S \text{ is } T_b^4 - AG - groupoid)$

Now let, $(ab.c)d = dc.ab \quad (by\ left\ invertive\ law)$

$\Rightarrow (ab.c)(ab) = dc.d \quad (S \text{ is } T_f^4 - AG - groupoid$

$\Rightarrow \quad d(ab.c) = ab.dc \quad (S \text{ is } T_b^4 - AG - groupoid)$

$= (dc.b)a \quad (by\ left\ invertive\ law)$

$\Rightarrow \quad d.a = (dc.b)(ab.c) \quad (S \text{ is } T_f^4 - AG - groupoid)$

$\Rightarrow (ab.c)d = a(dc.b) \quad (S \text{ is } T_b^4 - AG - groupoid)$

$= a(bc.d) \quad (by\ left\ invertive\ law)$

$\Rightarrow (ab.c)d = a(bc.d).$

Hence S is Bol* -AG-groupoid.

Since each Bol*- AG-groupoid is parmedial[4] and thus is left nuclear square[4], whence using Theorem 6 we immediately have the following:

Corollary 6: *Every T^4-AG-groupoid is paramedial AG-groupoid.*

Corollary 7: *Every T^4-AG-groupoid is left nuclear square AG-groupoid.*

Next we prove that the class of transitively commutative AG-groupoids contains the class of all T^4-*AG-groupoid*s.

Theorem 7: *Every T^4-AG-groupoid is transitively commutative AG-groupoid.*

Proof. Let S be T^4-AG-groupoid. Then $\forall\, a, b, c, d \in S,$ we have

$ab = cd \Rightarrow ad = cb \quad (S \text{ is } T_f^4 - AG - groupoid)$, and

$ab = cd \Rightarrow da = bc \quad (S \text{ is } T_b^4 - AG - groupoid)$

Now let, $ab = ba$ and $bc = cb.$

Consider $aa = bb$ and $bb = cc \Rightarrow aa = cc.$

Applying definition of T^4-AG-groupoid, we have,

$ac = ca.$

Hence S is transitively commutative AG-groupoid.

It is known that every T^1-AG-groupoid is AG^{**}-groupoid[11]. Here we prove that every cancellative AG^{**}-groupoid is T^1-AG-groupoid.

Theorem 8: *Every cancellative AG^{**}-*groupoid *is T^1-AG-groupoid.*

Proof: Let S be a cancellative AG^{**}-groupoid *and let x be a cancellative element of S.* Then $\forall\, a, b, c, d \in S,$ let $ab = cd.$

Now since,

$x^2(ba) = b(x^2a) \quad (S \text{ is } AG^{**} - groupoid)$

$= b(ax.x) \quad (by\ left\ invertive\ law)$

$= ax.bx \quad (S \text{ is } AG^{**} - groupoid)$

$= ab.xx \quad (by\ medial\ law)$

$= cd.xx \quad (by\ assumption)$

$= x(cd.x) \quad (S \text{ is } AG^{**} - groupoid)$

$= x(xd.c) \quad (by\ left\ invertive\ law)$

$= xd.xc \quad (S \text{ is } AG^{**} - groupoid)$

$= xx.dc \quad (by\ medial\ law)$

$x^2.ba = x^2.dc$

$\Rightarrow ba = dc \quad (S \text{ is cancellative})$

Hence S is T^1-AG-groupoid.

Conclusion

Many new classes of AG-groupoids have been discovered recently. Enumeration has also been done of these new classes up to order 6. All this has attracted researchers of the field to investigate these newly discovered classes in detail. This current article investigates the ideas of T^1, T^2 and T^4-AG-groupoids. We investigate that every T^4-AG-groupoid is Bol*-AG-groupoid. We further investigate that T^1 and T^4-AG-groupoids are paramedial and hence are left nuclear square. We also prove that T^1 and T^4 are transitively commutative AG-groupoid and T^1-AG-3-band is a semigroup.

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