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Application of Homotopy Analysis Method to SIR Epidemic Model

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Abstract

In this article, the problem of the spread of a non-fatal disease in a population which is assumed to have constant size over the period of the epidemic is considered. Mathematical modeling of the problem leads to a system of nonlinear ordinary differential equations. Homotopy analysis method is employed to solve this system of nonlinear ordinary differential equations.

Keywords: Homotopy Analysis method, System of non-linear differential equations; PACS: 02.60.Lj, 02.60.Cb, 87.10.Ed

Introduction

In 1992, Liao[1,2] proposed a new analytical technique; namely the Homotopy Analysis Method (HAM) based on homotopy of topology. However, in Liao's PhD dissertation[1], he did not introduce the auxiliary parameter $\hbar$, but simply followed the traditional concept of homotopy to construct the following one-parameter family of equations

$$(1-p)\mathsf{L}(u)+p\mathsf{N}(u)=0, \tag{1}$$

where L is an auxiliary linear operator, N is a nonlinear operator related to the original nonlinear problem $\mathsf{N}(u)=0$ and p is the embedding parameter. An improved two parameters family of equations was proposed to avoid divergence of solution by introducing an auxiliary parameter $\hbar$[3, 4].

$$(1-p)\mathsf{L}(u-u_0)=\hbar p\mathsf{N}(u). \tag{2}$$

where u_0 is an initial guess. Using the definition of Taylor series with respect to the embedding parameter p (which is a power series of p), Liao gave a general equations for high-order approximations. He[5, 6] constructed the one-parameter family of equations

$$(1-p)\mathsf{L}(u)+p\mathsf{N}(u)=0, \tag{3}$$

which is exactly the same as Liao's early one-parameter family equation (1), and is a special case of Liao's modified two-parameter equation (2) when $\hbar=-1$.

The problem of spreading of a non-fatal disease in a population which is assumed to have constant size over the period of the epidemic is considered in[7].

At time t suppose the population consist of

$x(t)$ susceptible: those so far uninfected and therefore liable to infection; $y(t)$ infective: those who have the disease and are still at large; $z(t)$ who are isolated, or who have recovered and are therefore immune.

Assume there is a steady constant rate between susceptible and infective and that a constant proportion of these constant result in transmission. Then in time δt, δx of the susceptible become infective, where

$$\delta x=-\beta xy\,\delta t \tag{4}$$

and β is a positive constant. If $\gamma>0$ is the rate at which current infective become isolated, then

$$\delta y=\beta xy\,\delta t-\gamma y\,\delta t. \tag{5}$$

The number of new isolates δz is given by

$$\delta z=\gamma y\,\delta t. \tag{6}$$

Now let $\delta t\to\infty$. Then the following system determines the progress of the disease:

$$\begin{cases} \dfrac{dx}{dt} = -\beta xy \\ \dfrac{dy}{dt} = \beta xy - \gamma y \\ \dfrac{dz}{dt} = \gamma y \end{cases} \quad (7)$$

with initial conditions,

$$x(0) = N_1, \quad y(0) = N_2 \quad z(0) = N_3 .$$

Homotopy Analysis Method (HAM)

In this paper we apply the homotopy analysis method to the problem. Consider the following equation:

$$\mathrm{N}[\vartheta(t)] = 0$$

where N is a nonlinear operator, t denotes the independent variable, $\vartheta(t)$ is an unknown function. The so-called HAM's zero-order deformation[8] can be obtained

$$(1-p)\mathrm{L}[\phi(t;p) - \vartheta_0^{[0]}(t)] = p\hbar H(t)\mathrm{N}[\vartheta(t)] \quad (8)$$

where $p \in [0,1]$ is the embedding parameter, $\hbar \neq 0$ is a nonzero auxiliary parameter, $H(t) \neq 0$ is an auxiliary function, L is an auxiliary linear operator, $\vartheta_0^{[0]}(t)$ is an initial guess of $\vartheta(t)$, $\phi(t;p)$ is an unknown function. When $p = 0$ and $p = 1$, then

$$\phi(t;0) = \vartheta_0^{[0]}(t), \qquad \phi(t;1) = \vartheta(t),$$

respectively. Thus, as p increases from 0 to 1, the solution $\phi(t;p)$ varies from the initial guess $\vartheta_0^{[0]}(t)$ to the solution $\vartheta(t)$. Expanding $\phi(t;p)$ in Taylor series with respect to p, one has

$$\phi(t;p) = \vartheta_0^{[0]}(t) + \sum_{k=1}^{+\infty} \vartheta_0^{[k]}(t)\, p^k , \quad (9)$$

where

$$\vartheta_0^{[k]}(t) = \frac{1}{k!} \frac{\partial^k \phi(t;p)}{\partial p^k}\bigg|_{p=0} . \quad (10)$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter $\hbar$, and the auxiliary function are properly chosen, the series (9) converges at $p = 1$, thus

$$\vartheta(t) = \vartheta_0^{[0]}(t) + \sum_{k=1}^{+\infty} \vartheta_0^{[k]}(t), \quad (11)$$

which must be one of the solutions of the original nonlinear equation, as proved by Liao 8.

As pointed by Liao[9], the auxiliary parameter $\hbar$ can be employed to adjust the convergence region of homotopy analysis solution. In general, by means of the so-called $\hbar$-curve, it is straightforward to choose an appropriate range for $\hbar$ which ensures the convergence of the solution series.

According to the equation (10), the governing equation can be deduced from the zero-order deformation equation (8). Define the vector

$$\vec{\vartheta}_m = \left\{ \vartheta_0^{[0]}(t), \vartheta_0^{[1]}(t), \cdots, \vartheta_0^{[m]}(t) \right\}.$$

Differentiating equation (8) k times with respect to the embedding parameter p and then setting $p = 0$ and finally dividing them by $k!$, we have the so-called kth-order deformation equation

$$\mathrm{L}[\vartheta_0^{[k]}(t) - \chi_k \vartheta_0^{[k-1]}(t)] = \hbar H(t) R_k(\vec{\vartheta}_{k-1}), \quad (12)$$

where

$$R_k(\vec{\vartheta}_{k-1}) = \frac{1}{(k-1)!} \frac{\partial^{k-1} \mathrm{N}[\phi(t;p)]}{\partial p^{k-1}}\bigg|_{p=0} , \quad (13)$$

and

$$\chi_k = \begin{cases} 0, & k \leq 1, \\ 1, & k > 1. \end{cases} \quad (14)$$

It should be emphasized that $\vartheta_0^{[k]}(t)$ for $k \geq 1$ is governed by the linear equation (10) with the linear boundary conditions that comes from the original problem, which can be solved easily by symbolic computation software such as Matlab, Maple or Mathematica. In this paper all calculations were accomplished using Maple software where the long format and the double precision have been used for high accuracy results.

Solution of the epidemic model by HAM

By applying inverse of the operator $\frac{d\,(.)}{dt}$, which is integration operator $\int_0^t (.)\, dt$ to each equations in the system (7) we derive

$$\begin{cases} x(t) = x(0) - \int_0^t \beta x(t)y(t)dt, \\ y(t) = y(0) + \int_0^t [\beta x(t)y(t) - \gamma y(t)]dt, \\ z(t) = z(0) + \int_0^t \gamma y(t)dt. \end{cases} \quad (15)$$

Let $x_0(t), y_0(t)$ and $z_0(t)$ are the initial guesses of $x(t), y(t)$ and $z(t)$ respectively.

According to equation (8), the HAM's zeroth-order deformation for equation (15) will be

$$\begin{cases} (1-p)(X(t,p) - x_0(t)) = \hbar p\left(X(t,p) - x(0) + \int_0^t \beta X(t,p)Y(t,p)dt \right) \quad (16) \\ (1-p)(Y(t,p) - y_0(t)) = \hbar p\left(Y(t,p) - y(0) - \int_0^t \beta[X(t,p)Y(t,p,\hbar) - \gamma Y(t,p)]dt \right) \\ (1-p)(Z(t,p) - z_0(t)) = \hbar p\left(Z(t,p) - z(0) - \int_0^t \gamma Y(t,p)dt \right) \end{cases}$$

For $p=0$ and $p=1$, we can write

$$\begin{cases} X(t,0) = x_0(t), X(t,1) = x(t) \\ Y(t,0) = y_0(t), Y(t,1) = y(t) \\ Z(t,0) = z_0(t), Z(t,1) = z(t) \end{cases} \quad (17)$$

Considering Maclaurin series of $X(t,p), Y(t,p)$ and $Z(t,p)$ corresponding to p, one has

$$\begin{cases} X(t,p) = x_0(t) + \sum_{k=1}^{+\infty} \frac{x_0^{[k]}(t)}{k!} p^k \\ Y(t,p) = y_0(t) + \sum_{k=1}^{+\infty} \frac{y_0^{[k]}(t)}{k!} p^k \\ Z(t,p) = z_0(t) + \sum_{k=1}^{+\infty} \frac{z_0^{[k]}(t)}{k!} p^k \end{cases} \quad (18)$$

which

$$x_0^{[k]}(t) = \left.\frac{\partial^k X(t,p)}{\partial p^k}\right|_{p=0}, \quad y_0^{[k]}(t) = \left.\frac{\partial^k Y(t,p)}{\partial p^k}\right|_{p=0}, \quad z_0^{[k]}(t) = \left.\frac{\partial^k Z(t,p)}{\partial p^k}\right|_{p=0}. \quad (19)$$

Define

$$x_k(t) = \frac{x_0^{[k]}(t)}{k!}, \quad y_k(t) = \frac{y_0^{[k]}(t)}{k!}, \quad z_k(t) = \frac{z_0^{[k]}(t)}{k!}. \quad (20)$$

If $p=1$, using equation (18), then

$$x(t) = x(0) + \sum_{k=1}^{+\infty} x_k(t), (t) = y(0) + \sum_{k=1}^{+\infty} y_k(t), (t) = z(0) + \sum_{k=1}^{+\infty} z_k(t). \quad (21)$$

Define the vectors

$$\begin{cases} \vec{x}_k = \{x_0(t), x_1(t), x_2(t), \ldots, x_k(t)\} \\ \vec{y}_k = \{y_0(t), y_1(t), y_2(t), \ldots, y_k(t)\} \\ \vec{z}_k = \{z_0(t), z_1(t), z_2(t), \ldots, z_k(t)\} \end{cases} \quad (22)$$

Thus we obtain the kth-order deformation equation

$$\begin{cases} \mathcal{L}[x_k(t) - \chi_k x_{k-1}(t)] = \hbar R_k(\vec{x}_{k-1}), \\ \mathcal{L}[y_k(t) - \chi_k y_{k-1}(t)] = \hbar R_k(\vec{y}_{k-1}), \\ \mathcal{L}[z_k(t) - \chi_k z_{k-1}(t)] = \hbar R_k(\vec{z}_{k-1}). \end{cases} \quad (23)$$

from equation (13) and (16), we have

$$\begin{cases} R_k(\vec{x}_{k-1}) = x_{k-1}(t) + \int_0^t \beta\left[\sum_{i=1}^{k-1} x_i(t) y_{k-1-i}(t)\right]dt - (x_0(t) - \chi_k x_0(t)), \\ R_k(\vec{y}_{k-1}) = y_{k-1}(t) - \int_0^t \left[\beta\left(\sum_{i=1}^{k-1} x_i(t) y_{k-1-i}(t)\right) - \gamma y_{k-1}(t)\right]dt - (y_0(t) - \chi_k y_0(t)), \\ R_k(\vec{z}_{k-1}) = z_{k-1}(t) - \int_0^t \gamma y_{k-1}(t)dt - (z_0(t) - \chi_k z_0(t)). \end{cases} \quad (24)$$

Considering equation (14), (16), (23) and (24) we can find the recursive expression of epidemic model

$$\begin{cases} x_k(t) = \chi_k x_{k-1}(t) + \hbar\left(x_{k-1}(t) + \int_0^t \beta\left[\sum_{i=1}^{k-1} x_i(t) y_{k-1-i}(t)\right]dt - (x_0(t) - \chi_k x_0(t))\right), \\ y_k(t) = \chi_k y_{k-1}(t) + \hbar\left(y_{k-1}(t) - \int_0^t \left[\beta\left(\sum_{i=1}^{k-1} x_i(t) y_{k-1-i}(t)\right) - \gamma y_{k-1}(t)\right]dt - (y_0(t) - \chi_k y_0(t))\right), \\ z_k(t) = \chi_k z_{k-1}(t) + \hbar\left(z_{k-1}(t) - \int_0^t \gamma y_{k-1}(t)dt - (z_0(t) - \chi_k z_0(t))\right). \end{cases} \quad (25)$$

Convergence Theorem: Theorem 3.1 If the solution series in equation (21) converge, where $x_k(t), y_k(t)$ and $z_k(t)$ is governed by equation (23) under definition (14) and (24), they must be the solution of equation (15).

Proof. If the series

$$\sum_{k=0}^{+\infty} x_k(t), \quad \sum_{k=0}^{+\infty} y_k(t), \quad \sum_{k=0}^{+\infty} z_k(t)$$

converge, we can write

$$X(t)=\sum_{k=0}^{+\infty} x_k(t), \quad Y(t)=\sum_{k=0}^{+\infty} y_k(t), \quad Z(t)=\sum_{k=0}^{+\infty} z_k(t) \tag{26}$$

and they hold

$$\lim_{k\to\infty} x_k(t)=0, \quad \lim_{k\to\infty} y_k(t)=0, \quad \lim_{k\to\infty} z_k(t)=0. \tag{27}$$

Using the definition of χ_k, we have

$$\begin{cases} \sum_{k=1}^{n}[x_k(t)-\chi_k x_{k-1}(t)] = x_1+(x_2-x_1)+(x_3-x_2)+\cdots+(x_n-x_{n-1})=x_n(t), \\ \sum_{k=1}^{n}[y_k(t)-\chi_k y_{k-1}(t)] = y_1+(y_2-y_1)+(y_3-y_2)+\cdots+(y_n-y_{n-1})=y_n(t), \\ \sum_{k=1}^{n}[z_k(t)-\chi_k z_{k-1}(t)] = z_1+(z_2-z_1)+(z_3-z_2)+\cdots+(z_n-z_{n-1})=z_n(t). \end{cases} \tag{28}$$

which gives us, according to (27)

$$\sum_{k=1}^{+\infty}[x_k(t)-\chi_k x_{k-1}(t)] = \lim_{k\to\infty} x_k(t)=0,$$

$$\sum_{k=1}^{+\infty}[y_k(t)-\chi_k y_{k-1}(t)] = \lim_{k\to\infty} y_k(t)=0, \tag{29}$$

$$\sum_{k=1}^{+\infty}[z_k(t)-\chi_k z_{k-1}(t)] = \lim_{k\to\infty} z_k(t)=0.$$

Furthermore, using the above expression and the definition of L, we have

$$\sum_{k=1}^{+\infty} L[x_k(t)-\chi_k x_{k-1}(t)] = L\sum_{k=1}^{+\infty}[x_k(t)-\chi_k x_{k-1}(t)]=0,$$

$$\sum_{k=1}^{+\infty} L[y_k(t)-\chi_k y_{k-1}(t)] = L\sum_{k=1}^{+\infty}[y_k(t)-\chi_k y_{k-1}(t)]=0, \tag{30}$$

$$\sum_{k=1}^{+\infty} L[z_k(t)-\chi_k z_{k-1}(t)] = L\sum_{k=1}^{+\infty}[z_k(t)-\chi_k z_{k-1}(t)]=0.$$

From the above expression and equation (23), we obtain

$$\sum_{k=1}^{+\infty} L[x_k(t)-\chi_k x_{k-1}(t)] = \hbar\sum_{k=1}^{+\infty} R_k(\vec{x}_{k-1})=0,$$

$$\sum_{k=1}^{+\infty} L[y_k(t)-\chi_k y_{k-1}(t)] = \hbar\sum_{k=1}^{+\infty} R_k(\vec{y}_{k-1})=0, \tag{31}$$

$$\sum_{k=1}^{+\infty} L[z_k(t)-\chi_k z_{k-1}(t)] = \hbar\sum_{k=1}^{+\infty} R_k(\vec{z}_{k-1})=0$$

which gives, since $\hbar \neq 0$, that

$$\sum_{k=1}^{+\infty} R_k(\vec{x}_{k-1})=0, \quad \sum_{k=1}^{+\infty} R_k(\vec{y}_{k-1})=0, \quad \sum_{k=1}^{+\infty} R_k(\vec{z}_{k-1})=0. \tag{32}$$

From (24), we have

$$\begin{aligned} \sum_{k=1}^{+\infty} R_k(\vec{x}_{k-1}) &= \sum_{k=1}^{+\infty}\left(x_{k-1}(t)+\int_0^t \beta\left[\sum_{i=0}^{k-1} x_i(t) y_{k-1-i}(t)\right]dt-(x_0(t)-\chi_k x_0(t))\right) \\ &= \sum_{k=1}^{+\infty} x_{k-1}(t)+\int_0^t \beta\left[\sum_{k=1}^{+\infty}\sum_{i=0}^{k-1} x_i(t) y_{k-1-i}(t)\right]dt - x_0(t) \\ &= \sum_{k=1}^{+\infty} x_{k-1}(t)+\int_0^t \beta\left[\sum_{i=1}^{+\infty} x_i(t)\sum_{j=0}^{+\infty} y_j(t)\right]dt - x_0(t) \\ &= X(t)+\int_0^t \beta X(t)Y(t)dt - x_0(t) \end{aligned} \tag{33}$$

From equations (32) and (33) we have

$$X(t)=x_0(t)-\int_0^t \beta X(t)Y(t)dt$$

Using similar procedure we obtain

$$Y(t)= y_0(t)+\int_0^t (\beta X(t)Y(t)-\gamma Y(t))dt$$

$$Z(t)=x_0(t)+\int_0^t \gamma Y(t)dt$$

Therefore, according to the above two equations, $X(t),(t)$ and $Z(t)$ must be exact solution of equation (15).

Numerical example

For numerical results the following values, for parameters, are considered: $N_1=20$, Initial population of $x(t)$, who are susceptible, $N_2=15$, Initial population of $y(t)$, who are infective, $N_3 = 10$, Initial population of $z(t)$, who are immune, $\beta = 0.01$, Rate of change of susceptible to infective population, $\gamma = 0.02$, Rate of change of infective to immune population.

Figures 1-3 show the $\hbar$-curves obtained from the 12th-order, 8th-order and 6th-order HAM approximation solutions of equation (21). From these figures, the valid regions of $\hbar$ correspond to the line segments nearly parallel to the horizontal axis.

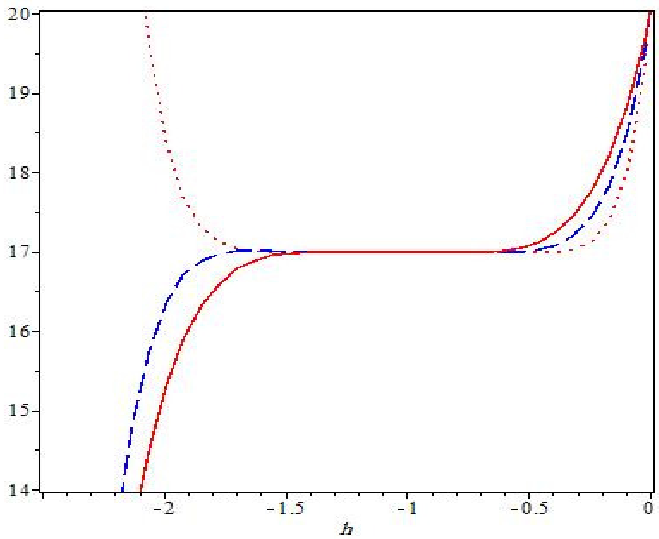

Figure-1
The $\hbar$-curve of $x(1)$ given by (21) Dotted line: 12th-order approximation; Dashed line: 8th-order approximation; Solid line: 6th-order

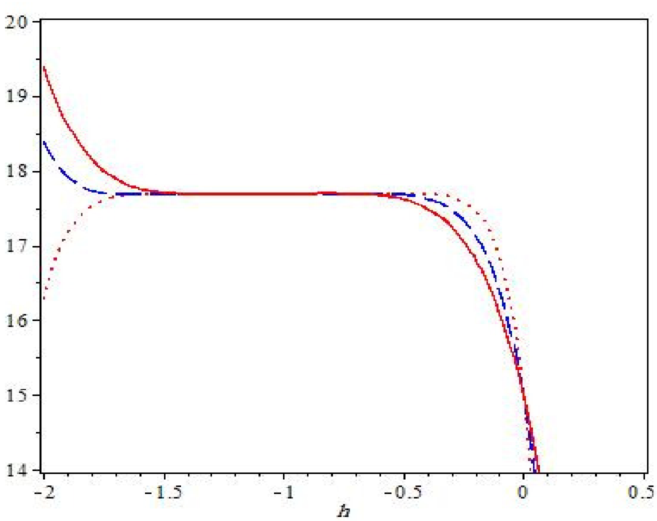

Figure-2
The $\hbar$-curve of $y(1)$ given by (21) Dotted line: 12th-order approximation; Dashed line: 8th-order approximation; Solid line: 6th-order

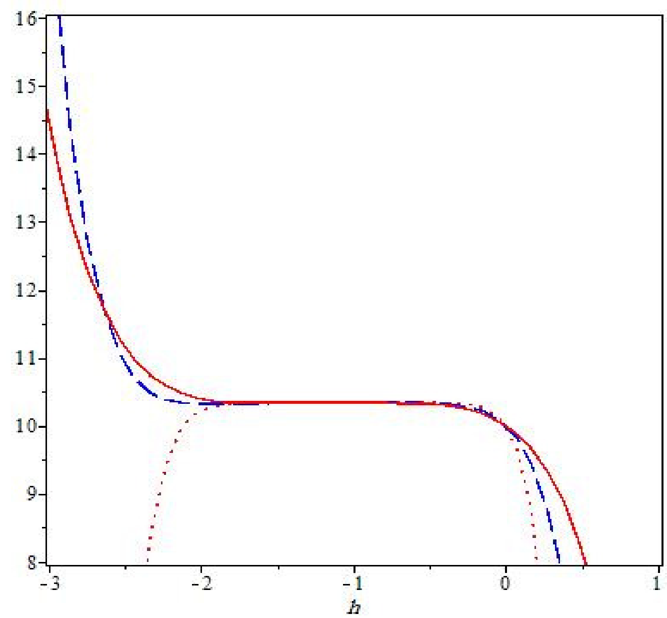

Figure-3
The $\hbar$-curve of $z(1)$ given by (21) Dotted line: 12th-order approximation; Dashed line: 8th-order approximation; Solid line: 6th-order

These results according to some values of $\hbar$ (the abscissa of intersection points in figures 1-3) are plotted in figures 4-6. As the plots show while the number of susceptible increases the population of who are infective decreases in the period of the epidemic, meanwhile the number of immune population increases.

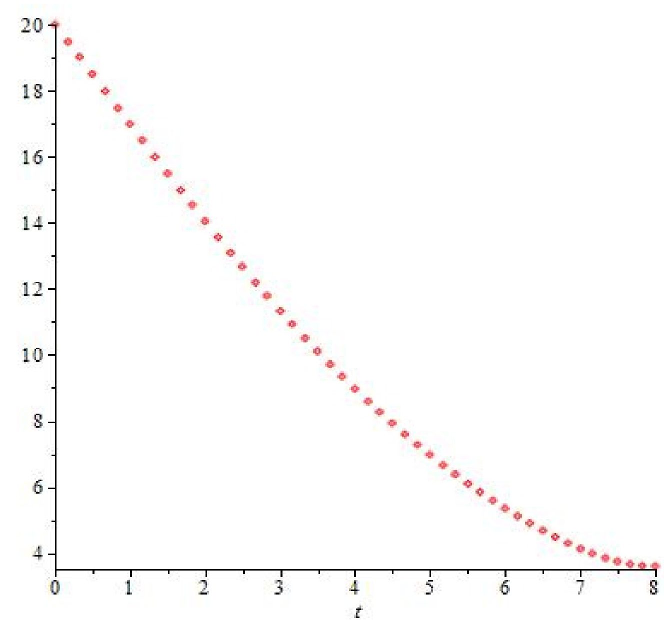

Figure-4
Plot of twelve terms approximation for $x(t)$ when
$\hbar = -1.0053219$

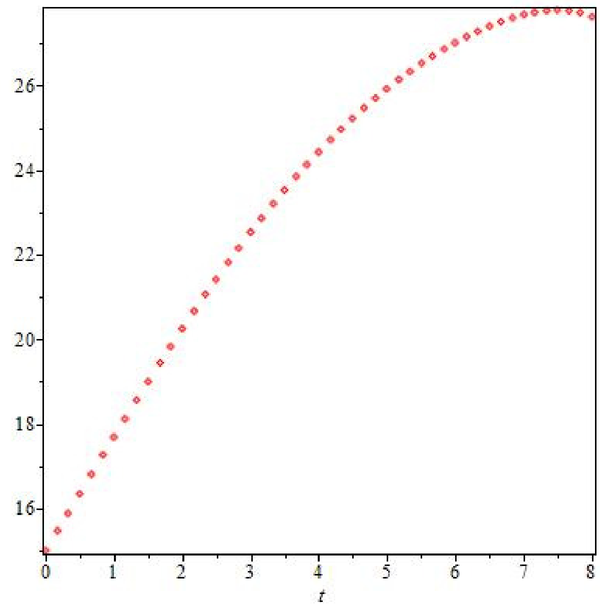

Figure-5

Plot of twelve terms approximation for $y(t)$ when $\hbar = -1.0047109$

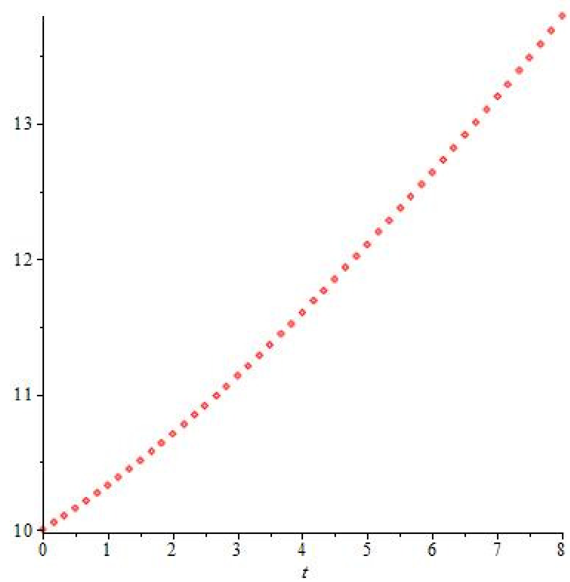

Figure-6

Plot of twelve terms approximation for $z(t)$ when $\hbar = -1.0639236$

Conclusion

Homotopy analysis method is known to be a powerful device for solving many functional equations such as ordinary, partial differential equations, integral equations and so many other equations. In this article, we used homotopy analysis method for solving a system of differential equation which describe SIR model for an epidemic disease. Using the concept of $\hbar$-curve, we found the convegence aera for the series solution of the problem. Numerical examples also provided to show the ability and efficiency of the method.

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