Research Journal of Recent Sciences ISSN 2277-2502
Vol. **2(1),** 67-70, January **(2013)** *Res.J.Recent Sci.*

Review Paper

On Introduction of New Classes of AG-groupoids

Shah M.[1], Ahmad I.[2] and Ali A.[1]
[1]Department of Mathematics, Quaid-e-Azam University, Islamabad, PAKISTAN
[2]Department of Mathematics, University of Malakand, Khyber Pakhtunkhwa, PAKISTAN

Available online at: www.isca.in
Received 12th August 2012, revised 8th September 2012, accepted 18th September 2012

Abstract

We discover eight new subclasses of AG-groupoids namely; Type-1, Type-2, Left Type-3, Right Type-3, Type-3, Backward Type-4, Forward Type-4, and Type-4. We provide examples of each type to prove their existence. We also give their enumeration up to order 6 and prove some of their basic properties and relations with each other and with other known classes.

Keywords: AG-groupoid, LA-group, AG-group, types of AG-groupoid, enumeration.

Introduction

A groupoid is called *AG-groupoid* if it satisfies the left invertive law[1]: $(ab)c = (cb)a$. An *AG^{**}-groupoid* is an AG-groupoid satisfying the identity $a(bc) = b(ac)$. An AG-groupoid with left identity is called *AG-monoid*. Every AG-monoid is AG^{**}-groupoid. An AG-groupoid S always satisfies the medial law[2; Lemma 1.1 (i)]: $(ab)(cd) = (ac)(bd)$ while an AG-monoid satisfies paramedial law[2; Lemma 1.2 (ii)]: $(ab)(cd) = (db)(ca)$. Note that the name *right modular groupoid*[2] is also used for AG-groupoid. An AG-groupoid S with left identity e is an AG^{**}-groupoid. An AG-groupoid S which satisfies $(ab)c = b(ac)$, for all $a,b,c \in S$, is called *AG^{*}-groupoid*. An AG-groupoid is called *Bol^{*}-groupoid* if it satisfies the identity $(ab \cdot c)d = a(bc \cdot d)$. An element a of an AG-groupoid S is called *idempotent* if $a^2 = a$. An AG-groupoid S is called *idempotent or AG-2-band or simply AG-band*[3] if its every element is idempotent. An AG-groupoid S is called *AG-3-band*[4] if its every element satisfies $a(aa) = (aa)a = a$. An AG-groupoid S is called *AG-group* if S contains left identity and inverses with respect to this identity. For detailed studies of this concept we refer the reader to reference[5, 6]. AG-groupoids (also called LA-semigroups), generalize commutative semigroups, have applications in flock theory[7] and some geometrical applications[5]. For additional sources on AG-groupoids, we suggest reference[8, 9] and for the semigroup concept we refer the reader to follow the book of Howie[10].

Recently we have discovered eight new interesting subclasses of AG-groupoid[11]. We introduce here more eight new subclasses of AG-groupoids which initially we call types. We give their counting up to order 6 and prove some relations between them and to other subclasses of AG-groupoids. We prove that every AG-3-band is T^1-AG-groupoid and T^2-AG-groupoid is T^3-AG-groupoid and every T^2-AG-groupoid is Bol^*-AG-groupoid. For T^1-AG-groupoid we prove that square of every element is idempotent and if it has left identity also then it becomes a unitary AG-group. As in semigroup theory the concept of zero-semigroup and zero-group exists, we find a similar concept for zero-AG-groupoid and zero-AG-group. Table-1 Presents the counting of new subclasses of AG-groupoids. Note that only the number of non-associative AG-groupoids is shown.

Table-1
Classification and enumeration results for new subclasses of AG- groupoids of orders 3–6

Order	3	4	5	6
Total number of AG-groupoids	20	331	31913	40104513
T^1-AG-groupoids	2	14	101	783
T^2-AG-groupoids	1	3	8	16
T_l^3-AG-groupoids	2	17	135	1272
T_r^3-AG-groupoids	3	36	374	5150
T^3-AG-groupoids	2	16	111	870
T_f^4-AG-groupoids	1	13	90	784
T_b^4-AG-groupoids	0	1	6	11
T^4-AG-groupoids	0	1	7	7

Type-1, Type-2, Type -3 and Type-4 AG-Groupoids

Definition 1: An AG-groupoid S is called a *Type-1 AG-groupoid* denoted by T^1-AG-groupoid if
$ab = cd \Rightarrow ba = dc,$ for all $a,b,c,d \in S.$
The following is now an obvious fact.

Proposition 1: Let S be an AG-groupoid. Then the following are equivalent: *i.* $ab = cd \Rightarrow ac = bd, \; \forall a,b,c,d \in S$; ii. $ab = cd \Rightarrow ca = db, \; \forall a,b,c,d \in S$.

Definition 2: An AG-groupoid S is called a *Type-2 AG-groupoid* denoted by T^2-AG-groupoid if $ab = cd \Rightarrow ac = bd, \; \forall a,b,c,d \in S$.

Definition 3: An AG-groupoid S is called a *Left Type-3 AG-groupoid* denoted by T_l^3-AG-groupoid if for all $a,b,c,d \in S$, $ab = ac \Rightarrow ba = ca.$

Definition 4: An AG-groupoid S is called a *Right Type-3 AG-groupoid* denoted by T_r^3-AG-groupoid if for all $a,b,c,d \in S$, $ba = ca \Rightarrow ab = ac.$

Definition 5: An AG-groupoid S is called a *Type-3 AG-groupoid* denoted by T^3-AG-groupoid if it is both T_l^3-AG-groupoid and T_r^3-groupoid.

Definition 6: An AG-groupoid S is called a *Forward Type-4 AG-groupoid* denoted by T_f^4-AG-groupoid if for all $a,b,c,d \in S$, $ab = cd \Rightarrow ad = cb$.

Definition 7: An AG-groupoid S is called a *Backward Type-4 AG-groupoid* denoted by T_b^4-AG-groupoid if $\forall a,b,c,d \in S$, $ab = cd \Rightarrow da = bc$.

Definition 8: An AG-groupoid S is called a *Type-4 AG-groupoid* denoted by T^4-AG-groupoid if it is both T_f^4-AG-groupoid and T_b^4-AG-groupoid.

Proposition 2: Let S be an AG-groupoid. Then S is a commutative semigroup if any of the following holds: i. $ab = cd \Rightarrow ad = bc, \forall a,b,c,d \in S,$ ii. $ab = cd \Rightarrow da = cb, \; \forall a,b,c,d \in S.$

Proof: Since $\forall a,b \in S$ the equation $ab = ab$ trivially holds. Now an application of either of *(i)* and *(ii)* proves commutativity in *S*. Since any commutative AG-groupoid *S* is associative, thus *S* becomes commutative semigroup.

There are some other cases but either they become semigroups or are identical to the cases that we have already considered. The following are examples or counter examples of some of the above considered types of AG-groupoid.

Example 1: *(i)* A T^3-AG-groupoid of order 3. *(ii)* A T^4-AG-groupoid of order 4 which is not T^2-AG-groupoid. *(iii)* A T^2-AG-groupoid of order 4 which is not T^4-AG-groupoid. *(iv)* A T^1-AG-groupoid of order 4 which is neither T^2-AG-groupoid nor T^4-AG-groupoid.

Let us first put the previous known facts involving these types into the new format. Thus the result[12; Theorem 2.7] now becomes:

Theorem 1: *Every AG-monoid is T^1-AG-groupoid.* Two generalizations of Theorem 1 have been considered by M. Shah[5] that can be read in the new scenario as:

Theorem 2: *Let S be an AG**-groupoid. Then S is a T^1-AG-groupoid if S has a cancellative element.* More generally,

Theorem 3: *Let S be an AG-groupoid. Then S is a T^1-AG-groupoid if S has a left invertive left cancellative element.* Regarding T^3-AG-groupoid the following fact is known.

Theorem 4: *Every AG-band is T^3-AG-groupoid*[3]. The following theorem generalizes the previous theorem to AG-3-band.

Theorem 5: *Every AG-3-band S is T^3-AG-groupoid*[3].

•	1	2	3
1	1	2	3
2	3	1	2
3	2	3	1

(i)

•	1	2	3	4
1	1	2	3	4
2	2	1	4	3
3	4	2	2	1
4	3	4	1	2

(ii)

•	1	2	3	4
1	1	2	3	4
2	1	1	3	4
3	4	4	1	3
4	3	3	4	1

(iii)

•	1	2	3	4
1	1	1	3	3
2	1	1	3	3
3	3	3	1	1
4	3	3	1	2

(iv)

Proof. Let $a,b,c \in S$. In order to prove that S is T_l^3-AG-groupoid let $ab = ac$. Then $ba = b^2b \cdot a = ab \cdot b^2 = ac \cdot b^2 = ab \cdot cb = ac \cdot cb = (aa^2)c \cdot cb = (ca^2)a \cdot cb = (ca^2)c \cdot ab = (ca^2)c \cdot ac = (ca^2)a \cdot c^2 = ac \cdot c^2 = ca$.
Now to prove that S is T_r^3–AG-groupoid, let $ba = ca$. Then

$$ab = a^2a \cdot b = ba \cdot a^2 = ca \cdot a^2 = ac.$$

Equivalently S is T^3-AG-groupoid.

An AG-group is said to be *unitary* if square of every element is equal to left identity.

Theorem 6: *Let S be a T^4-AG-groupoid. Then i. Square of every element of S is idempotent; ii. If S is an AG-monoid then S is a unitary AG-group.*

Proof: Let S be T^4-AG-groupoid. Then *i.* Obviously the identity $(aa)a = (aa)a$ holds trivially for every a in an AG-groupoid. Since S is a T^4-AG-groupoid, it becomes $(aa)a = a(aa)$. Hence S is locally associative. *ii.* Let S has left identity e then for all a in S we trivially have $ae \cdot e = ae \cdot e$, which by the property of T^4-AG-groupoid implies that $ae \cdot ae = ee$, which by medial law implies $a^2e = ee$, which then by cancellativity of e implies that $a^2 = e$. Hence the result.

Theorem 7: *Every T^1-AG-groupoid is Bol*-AG-groupoid.*

Proof: Let S be a T^1-AG-groupoid and $a,b,c \in S$. Then
$(ab \cdot c)d = dc \cdot ab,$ (by left invertive law)
$\Rightarrow d(ab \cdot c) = ab \cdot dc,$ (by definition of T^1-AG-groupoid)
$\Rightarrow d(ab \cdot c) = (dc \cdot b)a,$ (by left invertive law)
$\Rightarrow d(ab \cdot c) = (bc \cdot d)a,$ (by left invertive law)
$\Rightarrow (ab \cdot c)d = a(bc \cdot d),$ (by definition of T^1-AG-groupoid)

Hence the result.

Remark 1. The converse of the above theorem is not true as the Bol* AG-groupoid given in example 2 is not T^1-AG-groupoid.

Example 2. A Bol*-AG-groupoid.

•	1	2	3
1	1	1	1
2	1	1	1
3	1	2	2

From table 1 this is obvious that right Type-3-AG-groupoid is not necessarily left Type-3-AG-groupoid but one may guess the impression that the converse may be true. The following example shows that the converse is also false.

Example 3. A T_l^3-AG-groupoid of order 4 which is not T_r^3-AG-groupoid.

•	1	2	3	4
1	1	1	1	1
2	1	1	1	1
3	1	1	1	2
4	1	1	3	2

Theorem 8. *The following facts always hold, i. A T^2-AG-groupoid is T^1-AG-groupoid; ii. A T^4–AG-groupoid is T^1-AG-, iii. groupoid; iv. A T^1-AG-groupoid is, a. T_l^3-AG-groupoid, b. T_r^3-AG-groupoid; c. T^3-AG-groupoid.*

Proof: *i.* Let $a,b,c,d \in S$ and let $ab = cd$ which by definition of T^2-AG-groupoid implies that $ac = bd$. But then obviously $bd = ac$. Applying the same definition again we have $ba = dc$. Hence S is T^1-AG-groupoid. *ii.* Let $a,b,c,d \in S$ and let $ab = cd$ which by definition of T_f^4-AG-groupoid implies that $ad = cb$. Now applying definition of T_b^4-AG-groupoid we have $ba = dc$. Hence S is T^1-AG-groupoid. *iii. (a)* Apply definition of T^1-AG-groupoid with $a = c$, *b.* is similar to *(a)* and *(c)* follows from *(a)* and *(b).*

As a corollary we immediately have the following:

Corollary 1: *The following facts always hold, i. A T^2-AG-groupoid is, (a) T_l^3-AG-groupoid; (b) T_r^3-AG-groupoid; (c) T^3-AG-groupoid. (ii) A T^4-AG-groupoid is, (a) T_l^3-AG-groupoid; (b) T_r^3-AG-groupoid; (c) T^3-AG-groupoid.*

Zero-AG-Groupoid and Zero-AG-Group

As in the case of semigroups see for instance the book of Howie[10], there exists a zero-semigroup and zero-group, we prove the existence of Zero-AG-groupoid and zero-AG-group. Let us first define them.

Definition 9. An AG-groupoid *S* is called a *zero-AG-groupoid* if there exists an element z in *S* such that *S* without z is an AG-group and for all x in *S* we have that $xz = zx = z$.

Definition 10. An AG-groupoid *S* is called a *zero-AG-group* if there exists an element z in S such that S without z is a semigroup and for all x in S we have that $xz = zx = z$.

Now we provide some examples to show the existence of these concepts.

Example 4. *(i)* A zero-AG-groupoid of order 4. *(ii)* A zero-AG-group of order 3.

(i)

•	1	2	3	4
1	1	1	1	1
2	1	2	2	2
3	1	2	2	2
4	1	3	3	3

(ii)

•	1	2	3	4
1	1	1	1	1
2	1	2	3	4
3	1	4	21	31
4	1	3	42	2

Theorem 9: *Let* G *be an AG-group. Then* $Ga = aG = G, \forall a \in G.$

Proof: Clearly, $Ga \subseteq GG \subseteq G.$ Conversely, let $g \in G$ and let e be the left identity of G then, $g = eg = aa^{-1} \cdot g = ga^{-1} \cdot a \in Ga$. Therefore, $G \subseteq Ga.$ Hence $Ga \subseteq G.$ Next clearly $aG \subseteq GG \subseteq G.$ Conversely, let $g \in G$ then, $g = ee \cdot g = ge \cdot e = ge \cdot aa^{-1} = a(ge \cdot a^{-1}) \in aG.$ Therefore, $G \subseteq Ga.$ Hence $aG = G.$

Corollary 2: *Let* G *be an AG-group having left identity* e. *Then*[13] $G = eG = Ge$.

Corollary 3: *Let* G *be an AG-group. Then for all* $a, b \in G$, *there exist* $x, y \in G$ *such that*

$$ax = b, ya = b.$$

Proposition 3: *If an AG-groupoid S with 0 is a zero-AG-groupoid AG-group then* $\forall a \in S \setminus \{0\},\ Sa = aS = S$.

Proof: $S = G \cup \{0\}$ is a zero-AG-groupoid-AG-group where $G = S \setminus \{0\}$. Let $a \in S \setminus \{0\} \Rightarrow a \in G = S \setminus \{0\}$. As G is an AG-group, so by Theorem 9 $aG = Ga = G$. Now $aS = aG \cup \{0\} = G \cup \{0\} = S$, and $Sa = Ga \cup \{0\} = G \cup \{0\} = S$. Hence $Sa = aS = S$.

Conclusion

This article launches and investigates eight new classes of AG-groupoids. Enumeration of each class has also been done up to order 6. Relations of these newly discovered classes with each other and with previously known classes have been investigated to some extent. The readers are motivated to study these new classes in more detail.

References

1. Kazim M.A. and Naseerudin M., On almost semigroups, *Portugaliae Mathematica,* **36(1)**, (**1977)**
2. Cho J.R., Pusan Jezek J. and Kepka T., Praha, Paramedial Groupoids, *Czechoslovak Mathematical Journal*, **49(124)**, **(1996)** Praha
3. Stevanovic N. and Protic P.V., Abel-grassmann's bands, *Quasigroups and Related Systems*, **11(1)** 95–101 **2004**.
4. Stevanovic N. and Protic P.V., Composition of Abel-Grassmann's 3-bands, *Novi Sad J. Math*., **34(2),** 175−182 **(2004)**
5. Shah M., A theoretical and computational investigations of AG-groups, PhD thesis, Quaid-i-Azam University Islamabad, **(2012)**
6. Shah M. and Ali A., Some structural properties of AG-group, *International Mathematical Forum* 6, **34**, 1661–1667 **(2011)**
7. Naseeruddin M., Some studies on almost semigroups and flocks, Ph.D Thesis, The Aligarh Muslim University, India, (**1970)**
8. Mushtaq Q. and Yusuf S.M., On Locally Associative LA-semigroup, *J. Nat. Sci. Math.* Vol. XIX, No.1, 57–62 (**1979)**
9. Shah M., Shah T. and A. Ali, On the cancellativity of AG-groupoids, *International Mathematical Forum* 6, **44**, 2187–2194 **(2011)**
10. Howie J.M., Fundamentals of Semigroup Theory, *Clarendon Press, Oxford*, **(2003)**
11. Shah M., Ahmad I. and A. Ali, Discovery of new classes of AG-groupoids, *Res. J. Recent Sci.,* **1(11),** 47-49 **(2012)**
12. Mushtaq Q. and Yusuf S.M., On LA-semigroups, *Alig. Bull. Math*., **8(1),** 65-70 **(1978)**
13. Mushtaq Q. and Kamran M.S., On left almost groups, *Proc. Pak. Acad. of Sciences*, 33, 1–2 **(1996)**