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Some Construction Methods of Variance and Efficiency Balanced Block Designs with Repeated Blocks

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Abstract

Some construction methods of the variance and efficiency balanced block designs with repeated blocks are proposed which are based on the incidence matrices of the known balanced incomplete block designs with repeated blocks.

Keywords: Balance incomplete block design, balance incomplete block design with repeated blocks, variance balance design, efficiency balance design.

Introduction

The concept of repeated blocks came into existence in Van Lint[1]. He noticed that many of the BIB designs constructed by Hanani[2] have repeated blocks. In the early years, on the concept of repeated blocks, different type of work has been done by many statisticians, like Parker[3], Seiden[4], Stanton[5,6], Sprott [7,8], Ryser[9] etc. In 1973, Van Lint systematically studied the problems of the construction of BIB designs with repeated blocks. In the year 1977, Foody and Hedayat[10] presented some potential applications of the balanced incomplete block designs with repeated blocks. Designs with repeated blocks with the equireplications and with equal size of each block were discussed in the literature; Hedayat and Li[11], Hedayat and Hwang[12]. However from the practical point of view, it may not be possible to construct the design with equi block size accommodating the equireplication of each treatment in all the blocks. Here we consider a class of block designs called variance and efficiency balanced block designs which can be made available in unequal block sizes and for varying replications.

From the point of view of application there is no reason to exclude the possibility that a BIB design would contain repeated blocks. For a variety of reasons, it is desirable to have the balanced incomplete block designs with the block repetitions, because it might be less expensive and easier to implement. In many applications, the experimenter may not wish to run certain treatment combinations. For example, it is physically impossible to run three or more treatment combinations in one block. Thus we need BIB designs with repeated blocks. The set of all distinct blocks in a block design is called the support of the design and the cardinality of the support is denoted by b* and is referred to as the support size of the design.

It's not always necessary that every variance balanced design is also an efficiency balanced but in the present paper we proposed some construction methods of variance balanced block designs with repeated blocks which are also efficiency balanced with unequal block sizes and unequal replications. For this we take the reference of some construction methods of variance balanced and efficiency balanced block designs with repeated blocks given in research papers of Bronislaw Ceranka and Malgorzata Graczyk[13-15]. Using these construction schemes, which are based on the incidence matrices of the known BIB designs with repeated blocks; we proposed some new construction methods of variance and efficiency balanced designs with repeated blocks for ν treatments which are generalization of the schemes given in reference papers.

Let us consider ν treatments arranged in b blocks, such that the j^{th} block contains k_j. experimental units and the i^{th} treatment appears r_i times in the entire design, $i = 1,2,\ldots,\nu$; $j = 1,2,\ldots,b$. For any block design there exist a incidence matrix $N = [n_{ij}]$ of order ν x b, where n_{ij} denotes the number of experiment units in the j^{th} block getting the i^{th} treatment. When $n_{ij} = 1$ or 0 $\forall$ i and j, the design is said to be binary. Otherwise it is said to be nonbinary. In this paper we consider binary block designs only. The following additional notations are used $k = [k_1\ k_2 \ldots k_b]'$ is the column vector of block sizes, $r = [r_1\ r_2 \ldots r_v]'$ is the column vector of treatment replication, $K_{bxb} = \text{diag}\,[k_1\ k_2 \ldots k_b]$, $R_{vxv} = \text{diag}\,[r_1\ r_2 \ldots r_v]$, $\Sigma r_i = \Sigma k_j = n$ is the total number of experimental units , with this $N1_b = \underline{r}$ and $N'1_v = \underline{k}$, Where 1_a is the a x 1 vector of ones.

The information matrix for treatment effects C defined below as

$$C = R - NK^{-1}N' \qquad (1)$$

Where $R = \text{diag}\,(r_1, r_2 \ldots r_v)$, $K = \text{diag}\,(k_1, k_2 \ldots k_b)$

Though there have been balanced designs in various sense (see Puri and Nigam[16], Cali´nski[17], we will consider a balanced design of the following type. A block design is said to be balanced if every elementary contrast of treatment is estimated with the same variance (Rao[18]). In this sense this design is also called a variance balance design.

It is well known that block design is a variance balanced if and only if it has

$$C = \eta \left(I_v - \frac{1}{v} 1_v 1_v' \right) \quad (2)$$

where η is the unique nonzero eigenvalue of the matrix C with the multiplicity $v - 1$, I_v is the $v \times v$ identity matrix. For binary block design

$$\eta = \frac{\sum_{i=1}^{v} r_i - b}{v-1} \quad (3)$$

(Kageyama and Tsuji[19])

In particular case when block design is a balanced incomplete block design then $\eta = \frac{v r - b}{v-1}$.

A block design is called efficiency balanced if every contrast of treatment effects is estimated through the design with the same efficiency factor.

Let us consider the matrix M_o given by Cali´nski[20]

$$M_o = R^{-1}NK^{-1}N' - \frac{1}{n} 1_v r' \quad (4)$$

$M_o S = \mu S$

Where $T = [T_1\ T_2 ....T_v]'$ is the vector of treatment totals ; T_i is the total yield for the i^{th} treatment. μ is the unique non zero eigen value of M_o with multiplicity $(v-1)$ and M_o is given as (4).

Cali´nski[20]showed that for such designs every treatment contrast is estimated with the same efficiency $(1-\mu)$ and N is a EB block design if and only if

$$M_o = \mu \left(I_v - \frac{1}{n} 1_v r'\right) \quad (5)$$

Kageyama[21] proved that for the EB block design N, eq^n (5) is fulfilled if and only if

$$C = (1-\mu)\left(R - \frac{1}{n} r r'\right) \quad (6)$$

Construction for v treatments

Let N_i, i = 1, 2, ..., t, be the incidence matrix of the balanced incomplete block design with repeated blocks with the parameters v, b_i, r_i, k_i, λ_i, b_i^*. Let C_i be the C-matrix of this design defined by N_i, i = 1, 2,.....,t. Now, we form the matrix N as

$$N = [N_1\ N_2 \ ......\ N_t] \quad (7)$$

and prove the following theorem.

Theorem 1 : Block design with the incidence matrix N of the form (7) is the variance and efficiency balanced block design with repeated blocks with the parameters

$$v,\ b = \sum_{i=1}^{t} b_i,\quad r = \sum_{i=1}^{t} r_i,$$

$$k = \left[k_1 1'_{b_1} \quad k_2 1'_{b_2} \quad \cdots \quad k_t 1'_{b_t} \right]',$$

$$\lambda = \sum_{i=1}^{t} \lambda_i \ \text{ and } \ b^* = \sum_{i=1}^{t} b_i^* .$$

Proof : The matrix C of the block design (7) is

$$C = r I_v - \sum_{i=1}^{t} \frac{1}{k_i} N_i N_i'$$

$$C = \sum_{i=1}^{t} \left[\left(r_i - \frac{1}{k_i}(r_i - \lambda_i) \right) I_v - \frac{\lambda_i}{k_i} 1_v 1_v' \right]$$

$$C = \sum_{i=1}^{t} C_i = \sum_{i=1}^{t} \eta_i \left[I_v - \frac{1}{v} 1_v 1_v' \right] = \eta \left[I_v - \frac{1}{v} 1_v 1_v' \right]$$

where $\eta = \sum_{i=1}^{t} \eta_i$, η_i is the unique nonzero eigenvalue of the matrix C_i, $i = 1,2,........,t$. So, the theorem is proved and the Variance and Efficiency is given as,

$$\text{Variance} = v \sum_{i=1}^{t} \left(\frac{\lambda_i}{k_i} \right)$$

$$\text{Efficiency} = 1 - \frac{v}{r} \sum_{i=1}^{t} \left(\frac{\lambda_i}{k_i} \right) = \frac{1}{r} \sum_{i=1}^{t} \left(\frac{r_i - \lambda_i}{k_i} \right)$$

Example 1 : Let us consider the balanced incomplete block design with the parameters $v = 7$, $b_1 = 21$, $r_1 = 6$, $k_1 = 2$, $\lambda_1 = 1$, $b_1^* = 21$ with the incidence matrix N_1 given through the blocks (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (1, 7), (2, 3), (2, 4), (2, 5), (2, 6), (2, 7), (3, 4), (3, 5), (3, 6), (3, 7), (4, 5), (4, 6), (4, 7), (5, 6), (5, 7), (6, 7) and the balanced incomplete block design with the parameters $v = 7$, $b_2 = 14$, $r_2 = 6$, $k_2 = 3$, $\lambda_2 = 2$, $b_2^* = 7$ with the incidence matrix N_2 given through the blocks (1, 2, 4), (2, 3, 5), (3, 4, 6), (4, 5, 7), (1, 5, 6), (2, 6, 7), (1, 3, 7), each block is repeated two times. Based on the matrices N_1 and N_2 for t = 2 we form the incidence matrix N in the form (7) of the variance and efficiency balanced block design with repeated blocks with the parameters $v = 7$, $b = 21+14=35$, $r = 6+6=12$, $\lambda = 1+2 = 3$, $b^* = 21+7=28$ and $k = \left[2 1'_{21} \quad 3 1'_{14} \right]'$ and the structure is variance and efficiency balanced.

Hence the matrix C is given as $C = \frac{49}{6}\left[I_7 - \frac{1}{7} 1_7 1_7' \right]$ and

$$\text{Variance} = 7\left(\frac{1}{2} + \frac{2}{3} \right) = 49/6 = 8.1667$$

$$\text{Efficiency} = 1 - \frac{7}{12}\left(\frac{1}{2} + \frac{2}{3} \right) = 23/72 = 0.319444$$

Some New Construction methods of Variance and Efficiency Balanced Block Designs with repeated blocks

Theorem 2: Let N_1 be the incidence matrix of the balanced incomplete block design with repeated blocks with the parameters ν, b_1, r_1, k_1, λ_1, b_1^*. Then the Block design with the incidence matrix N of the form

$$N = [\, N_1 \;\; 1_v \,] \tag{8}$$

is the variance and efficiency balanced block design with repeated blocks with the parameters ν, $b = b_1 + 1$, $r = r_1 + 1$, $\lambda = \lambda_1 + 1$, $b^* = b_1^* + 1$, $k = \left[k_1 . 1'_{b_1} \;\; v \right]$ and the variance and efficiency is given as ,

$$\text{Variance} = 1 + v\left(\frac{\lambda_1}{k_1}\right)$$

$$\text{Efficiency} = 1 - \frac{1}{r}\left\{1 + v\left(\frac{\lambda_1}{k_1}\right)\right\} = \frac{1}{r}\left(\frac{r_1 - \lambda_1}{k_1}\right)$$

Theorem 3 : Let N_1 and N_2 be the incidence matrices of the balanced incomplete block design with repeated blocks with the parameters ν, b_1, r_1, k_1, λ_1, b_1^* and ν, b_2, r_2, k_2, λ_2, b_2^* respectively. Then the block design with the incidence matrix N of the form

$$N = [\, N_1 \; N_2 \; 1_v \,] \tag{9}$$

is the variance and efficiency balanced block design with repeated blocks with the parameters ν, $b = b_1 + b_2 + 1$, $r = r_1 + r_2 + 1$, $\lambda = \lambda_1 + \lambda_2 + 1$, $b^* = b_1^* + b_2^* + 1$, $k = \left[k_1 . 1'_{b_1} \;\; k_2 . 1'_{b_2} \;\; v \right]$ and the variance and efficiency is given as,

$$\text{Variance} = 1 + v\left(\frac{\lambda_1}{k_1} + \frac{\lambda_2}{k_2}\right)$$

$$\text{Efficiency} = 1 - \frac{1}{r}\left\{1 + v\left(\frac{\lambda_1}{k_1} + \frac{\lambda_2}{k_2}\right)\right\} = \frac{1}{r}\left(\frac{r_1 - \lambda_1}{k_1} + \frac{r_2 - \lambda_2}{k_2}\right)$$

Theorem 4 : Let N_i, i = 1, 2, ..., t, be the incidence matrix of the balanced incomplete block design with repeated blocks with the parameters ν, b_i, r_i, k_i, λ_i, b_i^*. Then block design with the incidence matrix N of the form

$$N = [\, N_1 \;\; N_2 \; \ldots\ldots . \; N_t \;\; 1_v \,] \tag{10}$$

is the variance and efficiency balanced block design with repeated blocks with the parameters v,

$$b = \sum_{i=1}^{t} b_i + 1, \quad r = \sum_{i=1}^{t} r_i + 1, \quad \lambda = \sum_{i=1}^{t} \lambda_i + 1,$$

$$b^* = \sum_{i=1}^{t} b_i^* + 1, \quad k = \left[k_1 . 1'_{b_1} \;\; k_2 . 1'_{b_2} \;\; \cdots \;\; k_t . 1'_{b_t} \;\; v \right]'$$

and the variance and efficiency is given as,

$$\text{Variance} = 1 + v\sum_{i=1}^{t} \frac{\lambda_i}{k_i}$$

$$\text{Efficiency} = 1 - \frac{1}{r}\left\{1 + v\sum_{i=1}^{t}\left(\frac{\lambda_i}{k_i}\right)\right\} = \frac{1}{r}\sum_{i=1}^{t}\left(\frac{r_i - \lambda_i}{k_i}\right)$$

Theorem 5 : Let N_i be the incidence matrix of the balanced incomplete block design with repeated blocks with the parameters ν, b_i, r_i, k_i, λ_i, b_i^*. Then block design with the incidence matrix N of the form

$$N = \left[N_i \; \overbrace{1_v \; 1_v \cdots\cdots 1_v}^{p-times} \right] \tag{11}$$

is the variance and efficiency balanced block design with repeated blocks with the parameters v,

$$b = b_i + p, \quad r = r_i + p, \quad \lambda = \lambda_i + p,$$

$$k = \left[k_i . 1'_{b_i} \; \overbrace{v \;\; v \;\; \cdots\cdots \;\; v}^{p-times} \right]' \text{ and } b^* = b_i^* + 1,$$

then the variance and efficiency is given as

$$\text{Variance} = \left(v\frac{\lambda_i}{k_i} + p \right)$$

$$\text{Efficiency} = \left[1 - \frac{v\lambda_i + pk_i}{r\,k_i} \right] = \left(\frac{r_i - \lambda_i}{r\,k_i} \right)$$

Theorem 6 : Let N_i be the incidence matrix of the balanced incomplete block design with repeated blocks with the parameters ν, b_i, r_i, k_i, λ_i, b_i^*. Then block design with the incidence matrix N of the form

$$N = \left[\overbrace{N_i \; N_i \cdots\cdots N_i}^{p-times} \;\; 1_v \right] \tag{12}$$

is the variance and efficiency balanced block design with repeated blocks with the parameters v, $b = pb_i + 1$,

$$r = pr_i + 1, \quad \lambda = p\lambda_i + 1, \quad b^* = b_i^* + 1$$

and $k = \left[\overbrace{k_i . 1'_{b_i} \; k_i . 1'_{b_i} \cdots\cdots k_i . 1'_{b_i}}^{p-times} \;\; v \right]'$

and the variance and efficiency is given as,

$$\text{Variance} = \left(1 + pv\frac{\lambda_i}{k_i} \right)$$

$$\text{Efficiency} = \left[1 - \frac{pv\lambda_i + k_i}{r\,k_i} \right] = \left[\frac{p(r_i - \lambda_i)}{r\,k_i} \right]$$

Theorem 7: Let N_i, i = 1, 2, ..., t, be the incidence matrix of the balanced incomplete block design with repeated blocks with the parameters ν, b_i, r_i, k_i, λ_i, b_i^*. Then block design with the incidence matrix N of the form

$$N = \left[N_1\, N_2 \cdots\cdots N_t \; \overbrace{1_v\; 1_v \cdots\cdots\cdots 1_v}^{p-times} \right] \quad (13)$$

is the variance and efficiency balanced block design with repeated blocks with the parameters v,

$$b = \sum_{i=1}^{t} b_i + p, \quad r = \sum_{i=1}^{t} r_i + p, \quad \lambda = \sum_{i=1}^{t} \lambda_i + p,$$

$$k = \left[k_1 1'_{b_1} \; k_2 1'_{b_2} \cdots\cdots k_t 1'_{b_t} \; \overbrace{v \quad v \quad \cdots\cdots\cdots \quad v}^{p-times} \right]',$$

$b^* = \sum_{i=1}^{t} b_i^* + 1$ and the variance and efficiency is given as,

$$\text{Variance} = \left[v \sum_{i=1}^{t} \left(\frac{\lambda_i}{k_i} \right) + p \right]$$

$$\text{Efficiency} = 1 - \frac{1}{r} \left\{ p + v \sum_{i=1}^{t} \left(\frac{\lambda_i}{k_i} \right) \right\} = \frac{1}{r} \sum_{i=1}^{t} \left(\frac{r_i - \lambda_i}{k_i} \right)$$

Conclusion

The construction methods discussed above are flexible enough to incorporate number of incidence matrices of balanced incomplete block design with repeated blocks. In the pattern given in (10), Variance increases but efficiency doesn't show any regular pattern. In the patterns given in (11) and (13), Variance increases and efficiency decreases as unit vector 1_v repeated number of times. In the pattern given in (12), Variance and efficiency both increases as incidence matrix N_i repeated number of times.

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