Abstract

In this paper we study η-Einstein α-Sasakian manifolds admitting Ricci soliton.

Keywords: Ricci soliton, α-Sasakian manifold, η-Einstein manifold.

Mathematical Subject Classification: 53C25, 53C21, 53C44.

Introduction

A Ricci soliton is a generalization of an Einstein metric. In a Riemannian manifold \((M, g)\), \(g\) is called a Ricci soliton studied by Hamilton\(^1\) if
\[
(\mathcal{L}_V g)(X, Y) + 2S(X, Y) + 2\lambda g(X, Y) = 0, \tag{1}
\]
where \(\mathcal{L}\) is the Lie derivative, \(S\) is the Ricci tensor, \(\lambda\) is a constant and \(V\) is a potential vector field on \(M\). Metrics satisfying (1) are interesting and useful in physics and are often referred as quasi-Einstein. Compact Ricci soliton are special case of the Ricci flow \(\frac{\partial}{\partial t} g_{ij} = -2S_{ij}\) with fixed point. There are many authors Perelman\(^2\) which study compact Ricci soliton and obtain many good results.

If \(\lambda\) is negative the Ricci soliton is said to be shrinking, if \(\lambda\) is zero the Ricci soliton is said to be steady and if \(\lambda\) is positive the Ricci soliton is said to be expanding. \(g\) is said to be a gradient Ricci soliton if the vector field \(V\) is the gradient of a potential function \(-f\) and the equation (1) has written of the form
\[
\mathcal{L}_V g = S + \lambda g.
\]
In dimension 2 and 3, a Ricci soliton on a Riemannian manifold is Sasakian manifold. Sasakian manifolds is a case of \(\alpha\)-Sasakian manifold. In section 3, we have a theorem and an example of a Sasakian-space form (generalized) \(M(f_1, f_2, f_3)\) with \(f_1 = (c + 3\alpha^2)/4\) and \(f_2 = f_3 = (c - \alpha^2)/4\). Also, it is η-Einstein, and follows all the conclusion of the theorem and \(M\) is \(R^{(2n+1)}(\alpha^2 - 4)\) recognizable with the \((2n + 1)\)-dimensional Heisenberg group.

α-Sasakian Manifolds

A contact manifold is a \((2n+1)\)-dimensional \(C^\infty\) manifold \(M\) equipped with a global form \(\eta\), called a contact form of \(M\) such that \(\eta \wedge (d\eta)^n \neq 0\) everywhere on \(M\). In particular, \(\eta \wedge (d\eta)^n \neq 0\) is a volume element of \(M\) so that a contact manifold is orientable. A contact manifold associated with the Riemannian metric \(g\) is called contact metric manifold if it satisfy the following relation (3)
\[
d\eta(X, Y) = g(X, \phi Y), \eta(X) = g(X, \xi), \quad \phi^2 = -I + \eta \delta \xi.
\]

If Ricci tensor of \(\alpha\)-Sasakian manifolds is written like (2) then it is η-Einstein manifold which is given as
\[
S(X, Y) = a g(X, Y) + b \eta(X)\eta(Y), \tag{2}
\]
where \(a\) and \(b\) are constant for \(n > 1\), Zhang\(^6\) studied compact Sasakian manifold with constant curvature and quasi-positive holomorphic bisectional transverse curvature. Sharma and Ghosh\(^7\) show that, if a 3-dimensional Sasakian metric is a non trivial Ricci soliton, then it is homothetic to the standard Sasakian structure on Heisenberg group nil\(^3\). A \(K\)-contact manifold is Sasakian manifold in dimension 3 which is not true in higher dimension.

This paper organised as follow:
both $\alpha$ and $\beta$ vanish, then $M$ is a cosymplectic manifold. Here we consider $\alpha$-Sasakian manifold and following holds in $\alpha$-Sasakian manifold,
\[
\nabla_X \xi = -\alpha \phi X, \quad (8)
\]
\[
\nabla(X, \xi) = \alpha (X - \eta(X)\xi). \quad (9)
\]
\[
Q_X = 2n\alpha, \quad (10)
\]
\[
(\nabla_X \phi)Y = \alpha (g(X, Y)\xi - \eta(Y)X), \quad (11)
\]

**Theorems and Example**

**Theorem:** If the $\eta$-Einstein (non-Einstein) $\alpha$-Sasakian manifold $M(\phi, \eta, \xi, g)$ has Ricci soliton (non-trivial) with potential vector $V$, then

(i). Jacobi along the geodesics is $V$ which is determined by $\xi$, then

\[
\nabla(X, \xi) \xi = \alpha (g(X, Y)\xi - \eta(Y)X), \quad (18)
\]

setting $Y = Z = \xi$ in (18) shows that,
\[
(\nabla_X \phi)Y = 4\alpha b \eta(X)\xi - X \xi + \eta(Y)(\nabla(X, \xi)\xi - \eta(Y)X), \quad (19)
\]

next Lie differentiation (9) along $V$ and using (19) and (13) we get,
\[
4\alpha b \eta(X)\xi - X \xi + \eta(Y)(\nabla(X, \xi)\xi - \eta(Y)X) = \alpha (-\eta(X)\nabla(X, \xi)\xi - g(X, \nabla(X, \xi)\xi) + 2(\lambda + a + b)\eta(X)\xi), \quad (20)
\]

next contracting (20) over $X$ and $g(\nabla(X, \xi)\xi - \eta(Y)X)$ gives
\[a - b + \lambda = 0, \quad (21)\]

now we use integrability condition of the Ricci soliton we get,
\[
\nabla r = -\Delta r + 2\lambda r + 2|S|^2, \quad (22)
\]

Where $\Delta r = -\text{div.Dr.}$ Comparing the value of $|S|^2$ from (2) and using (10) and (12) we find that $b(a + 2) = 0$. Since $b \neq 0$, because if $b = 0$ then $M$ is Einstein which is a contradiction hence we have,
\[a = -2 \text{ and } b = 2(n + 1). \]

Thus, it follows that $\lambda = 2(n + 2) > 0$, which show that Ricci soliton is expanding, which prove part (ii) of the theorem.

Contracting (18) along $X$ and using the formula (div$\phi$)$X = -2n\eta(X)$ for a contact metric one gets
\[
(\nabla_X \phi)Y = 4\alpha b \eta(X)\xi - X \xi + \eta(Y)(\nabla(X, \xi)\xi - \eta(Y)X), \quad (23)
\]

next, in (2) we take the Lie-derivative of $S(X, Y)$ along $V$ and then using (13) we get,
\[
(\nabla_X \phi)Y = -2(\alpha^2 + a\lambda)g(Y, Z) + b(\nabla\eta)Y)\eta(Z) + \eta(Y)(\nabla\eta)Z - 2ab\eta(Y)\eta(Z). \quad (24)
\]

comparing above two equations and put $Z = \xi$, and substituting the value of $a, b$ and $\lambda$ obtained above, we get $\nabla \eta = -4(n + \alpha)$, $V$ is infinitesimal contact transformation which depends on the value of $\alpha$, which is the part (iii) of the theorem. Also by the straightforward calculation, we find that $\xi = 4(n + \alpha)\xi$. Thus proof of the theorem is complete.

**Example**

A $M(f_1, f_2, f_3)$ generalized Sasakian-space-form which is $\alpha$-Sasakian manifold with $f_1 = (c + 3\alpha^2)/4$ and $f_2 = f_3 = (c - \alpha^2)/4$. Also, it is $\eta$-Einstein hence it follow the theorem. The value of $a$ and $b$ for generalized Sasakian-space-form are $a = \frac{n(c + 3\alpha^2) + (c - \alpha^2)}{2}$ and $b = \frac{(c - \alpha^2)(n + 1)}{2}$. Now from these
values, comparing the values of \(a\) and \(b\) which get from the theorem, we get \(c = a^2 - 4\). Thus \(M(f, f, f)\) is \(\mathbb{R}^{2n+1}(a^2 - 4)\) identifiable with the \((2n + 1)\)-dimensional Heisenberg group. This prove the corollary. Hence \(M\) is \(\mathbb{R}(\mathbb{C})/(\alpha - 4)\) recognizable with the \((2n + 1)\)-dimensional Heisenberg group.

**Conclusion**

In this paper we study \(\alpha\)-Sasakian manifold whose metric manifolds whose metric as Ricci soliton and we can see that when it is non-trivial Ricci soliton with potential vector \(V\) then Ricci soliton is expanding. \(V\) is Jacobi along geodesics determine by \(\xi\) and \(V\) is infinitesimal contact transformation.

**Acknowledgement**

The author Ankita Rai is supported by UGC (University Grant Commission) (JRF) fellowship for her research work.

**References**