Some results for Monotone and Nondecreasing Operators in Partial Metric spaces

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Abstract
In ordered partial metric spaces we introduced some fixed point results, without any type of commutativity of the concerned maps, we established coupled coincidence results. Also we give some results for nondecreasing mappings.

Keywords: Partial metric space; coupled fixed point; coupled coincidence point; nondecreasing mappings.

Introduction
In 1994 the concept of partial metric spaces was introduced by Matthews¹. This concept is introduced to give a modified version of the Banach contraction principle²,³. The existence and uniqueness of a fixed point of different contractive conditions for mappings satisfying on partial metric spaces⁴,⁵ was studied by several authors. In this paper we extend Luong and Thuan,⁶ results. O’Regan D, Petruşel A⁷ gave some existence results for Fredholm and Volterra type integral equations. In some of their works, the fixed point result is also given for nondecreasing mappings.

Preliminaries

Definition-1
A partial metric on a nonempty set X is a function $p : X \times X \to \mathbb{R}^+$ such that for all $x, y, z \in X$

p1. $x = y \iff p(x, x) = p(x, y) = p(y, y)$

p2. $p(x, x) \leq p(x, y)$

p3. $p(x, y) = p(y, x)$

p4. $p(x, z) \leq p(x, y) + p(y, z) - p(y, y)$

Lemma 1: Let $(X, p)$ be a partial metric space. Then (a) $\{X_n\}$ is a Cauchy sequence in $(X, p)$ if and only if it is a Cauchy sequence in the metric space $(X, p)$.

(b) $(X, p)$ is complete if and only if the metric space $(X, p)$ is complete. Furthermore, $\lim_{n \to \infty} p(x_n, x) = 0$ if and only if $p(x, x) = \lim_{n \to \infty} p(x_n, x) = \lim_{n \to \infty} p(x_n, x_m)$.

Let $(X, p)$ be a partial metric. We endow the product space $X \times X$ with the partial metric $q$ defined as follows: for $(x, y), (u, v) \in X \times X$, $q((x, y), (u, v)) = p(x, u) + p(y, v)$.

A mapping $F : X \times X \to X$ is said to be continuous at $(x, y) \in X \times X$ if for each $\epsilon > 0$, there exists $\delta > 0$ such that $F(B_{\delta}((x, y), \delta)) \subseteq B_{\epsilon}((F(x, y), \epsilon))$.

Definition-2
Let $F : X \times X \to X$ be a mapping in a partially ordered set $(X, \leq)$ and $F$ has the mixed monotone property if for any $x, y, u, v \in X$,

$x \leq y \Rightarrow F(x, y) \leq F(u, v)$ and $y \leq x \Rightarrow F(x, y) \geq F(u, v)$.

Definition-3
The point $(x, y) \in X \times X$ is a coupled fixed point of $F : X \times X \to X$ if $F(x, y) = x$ and $F(y, x) = y$.

Main results

Theorem 1
Suppose the metric $p$ on partially ordered set $(X, \leq)$ and $(X, d)$ is a complete partial metric space. Let the mapping $F : X \times X \to X$ be a mapping in a partially ordered set $(X, \leq)$ and $F$ has the mixed monotone property if for any $x, y, u, v \in X$,

$x \leq y \Rightarrow F(x, y) \leq F(u, v)$ and $y \leq x \Rightarrow F(x, y) \geq F(u, v)$.

Definition 4
The mapping $F : X \times X \to X$ in partially ordered set $(X, \leq)$ has the mixed $g$-monotone property if for any $x, y, u, v \in X$,

$x \leq y \Rightarrow F(x, y) \leq F(u, v)$ and $y \leq x \Rightarrow F(x, y) \geq F(u, v)$.

Theorem 2
Suppose the metric $p$ on partially ordered set $(X, \leq)$ and $(X, d)$ is a complete partial metric space. Let the mapping $F : X \times X \to X$ having the mixed monotone property on $X$. Let $x_0, y_0 \in X$ and $x_0 \leq F(x_0, y_0)$ and $y_0 \leq F(y_0, x_0)$.

Suppose $\exists \phi \in \Phi$ and $\psi \in \Psi$ such that

$\phi(p(F(x, y), (F(u, v))) \leq \frac{1}{2} \phi(p(x, u) + p(y, v)) - \psi(p(x, u) + p(y, v))$ (1)

$\forall x, y, u, v \in X$ with $x \geq u$ and $y \leq v$. Suppose either (a) The mapping $F$ is continuous or (b) $X$ has the following property:
(i) If \( \{ x_n \} \) is a non-decreasing sequence such that \( \{ x_n \} \rightarrow x \), then \( x_n \leq x \) \( \forall \ n \).

(ii) If \( \{ y_n \} \) is a non-increasing sequence such that \( \{ y_n \} \rightarrow y \) then \( y \leq y_n \) \( \forall \ n \).

Then \( \exists \ x,y \in X \) such that \( x = F(x,y) \) and \( y = F(y,x) \), that is, \( F \) has a coupled fixed point in \( X \).

**Proof**

Let \( x_0, y_0 \in X \) be such that \( x_0 \leq F(x_0, y_0) \) and \( y_0 \geq F(y_0, x_0) \). We construct sequences \( \{ x_n \} \) and \( \{ y_n \} \) in \( X \) as
\[
x_{n+1} = F(x_n, y_n) \quad \text{and} \quad y_{n+1} = F(y_n, x_n) \quad \forall \ n \geq 0.
\]

We are to prove that
\[
x_n \leq x_{n+1} \quad \forall \ n \geq 0
\]
and
\[
y_n \geq y_{n+1} \quad \forall \ n \geq 0.
\]

By using mathematical induction method (3) and (4) hold \( \forall \ n \geq 0 \). Therefore, \( x_0 \leq x_1 \leq x_2 \leq \ldots \leq x_n \leq \ldots \)
and \( y_0 \geq y_1 \geq y_2 \geq \ldots \geq y_n \geq y_{n+1} \geq \ldots \)

Since \( x_n \geq x_{n-1} \) and \( y_n \leq y_{n-1} \) using (1) and (2) we get
\[
\phi(p(x_{n+1}, y_{n+1})) = \phi(p(F(x_n, y_n), F(x_{n-1}, y_{n-1}))) \leq \frac{1}{2} \phi(p(x_n, x_{n-1}) + p(y_n, y_{n-1})) - \frac{1}{2} \psi(p(x_n, y_n) + p(y_n, x_n))
\]

Similarly, since \( y_{n+1} \geq y_n \) and \( x_{n+1} \leq x_n \), using (1) and (2), we also have
\[
\phi(p(y_{n+1}, y_{n+1})) = \phi(p(F(y_n, x_n), F(y_{n-1}, x_{n-1}))) \leq \frac{1}{2} \phi(p(y_n, y_{n-1}) + p(x_n, x_{n-1})) - \frac{1}{2} \psi(p(x_n, y_n) + p(y_n, x_n))
\]

so we have
\[
\phi(p(x_{n+1}, x_n) + p(y_{n+1}, y_n)) \leq \phi(p(x_n, x_{n-1}) + p(y_n, y_{n-1})) - 2\psi\left(\frac{p(x_n, x_{n-1}) + p(y_n, y_{n-1})}{2}\right)
\]

By property (\( \phi \)), we have
\[
\phi(p(x_{n+1}, x_n) + p(y_{n+1}, y_n)) \leq \phi(p(x_n, x_{n-1}) + p(y_n, y_{n-1}))
\]

so we have
\[
\phi(p(x_{n+1}, x_n) + p(y_{n+1}, y_n)) \leq \phi(p(x_n, x_{n-1}) + p(y_n, y_{n-1})) - 2\psi\left(\frac{p(x_n, x_{n-1}) + p(y_n, y_{n-1})}{2}\right)
\]

Since \( \psi \) is a positive function, therefore
\[
\phi(p(x_{n+1}, x_n) + p(y_{n+1}, y_n)) \leq \phi(p(x_n, x_{n-1}) + p(y_n, y_{n-1}))
\]

Now we use the fact that \( \phi \) is increasing, we get
\[
p(x_{n+1}, x_n) + p(y_{n+1}, y_n) \leq p(x_n, x_{n-1}) + p(y_n, y_{n-1})
\]

Set \( \delta_n = p(x_{n+1}, x_n) + p(y_{n+1}, y_n) \)

clearly \( \{ \delta_n \} \) is decreasing. Therefore,
\[
\lim_{n \to \infty} \delta_n = \lim_{n \to \infty} [p(x_{n+1}, x_n) + p(y_{n+1}, y_n)] = \delta_n = 0 \quad \forall \ n \geq 0.
\]

It can be shown that \( \delta_n = 0 \) as \( n \to \infty \), that is,
\[
\lim_{n \to \infty} \delta_n = \lim_{n \to \infty} [p(x_{n+1}, x_n) + p(y_{n+1}, y_n)] = 0.
\]

Let \( \delta_n^* = p^+(x_n, x_{n+1}) + p^+(y_n, y_{n+1}) \quad \forall \ n \in N \).

By definition of \( p^+ \), clearly that \( \delta_n^* \leq \delta_n \) for all \( n \in N \). Using (6), we get
\[
\lim_{n \to \infty} \delta_n^* = \lim_{n \to \infty} [p^+(x_n, x_{n+1}) + p^+(y_n, y_{n+1})] = 0
\]

Now we will show that \( \{ x_n \} \) and \( \{ y_n \} \) are Cauchy sequences in \( (X, \rho) \). On the contrary we assume that at least one of \( \{ x_n \} \) and \( \{ y_n \} \) is not a Cauchy sequence. Then \( \exists \ \epsilon > 0 \) for which we can find subsequences \( \{ x_{n(k)} \}, \{ y_{n(k)} \} \) of \( \{ x_n \} \) and \( \{ y_n \} \), \( \{ y_{m(n)} \} \) of \( \{ y_n \} \) with \( m(n) > n(k) \geq k \) such that
\[
p^+(x_{n(k)}, x_{m(n)}) + p^+(y_{n(k)}, y_{m(n)}) \geq \epsilon.
\]

Now we take \( m(n) > n(k) \). Then
\[
p^+(x_{n(k)}, x_{m(n)}) + p^+(y_{n(k)}, y_{m(n)}) < \epsilon.
\]

by triangle inequality, we have
\[
\epsilon \leq \epsilon_k = p^+(x_{n(k)}, x_{m(k)}) + p^+(y_{n(k)}, y_{m(k)})
\]

\[
\leq p^+(x_{n(k)}, x_{m(k)}) + p^+(y_{n(k)}, y_{m(k)})
\]

\[
\leq p^+(x_{n(k)}, x_{m(k)}) + p^+(y_{n(k)}, y_{m(k)}) + \epsilon.
\]

Taking \( k \to \infty \) and by (6), we get
\[
\lim_{k \to \infty} \epsilon_k = \lim_{k \to \infty} [p^+(x_{n(k)}, x_{m(k)}) + p^+(y_{n(k)}, y_{m(k)})] = \epsilon.
\]

By the triangle inequality,
\[
\epsilon_k = p^+(x_{n(k)}, x_{m(k)}) + p^+(y_{n(k)}, y_{m(k)})
\]

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\[ \leq p^* \left( x_{m(k)} \right) + p^* \left( y_{m(k)} \right) \]

Then by definition of \( r_k \)
\[ r_k = p^* \left( x_{m(k)} \right) + p^* \left( y_{m(k)} \right) \]

Similarly, \( \phi \left( p^* \left( y_{m(k)} \right) \right) \leq \phi \left( r_k \right) - 2\psi \left( \frac{r_k}{2} \right) \)

So adding we get
\[ \phi \left( p^* \left( x_{m(k)} \right) \right) + \phi \left( p^* \left( y_{m(k)} \right) \right) \leq 2\phi \left( r_k \right) - 4\psi \left( \frac{r_k}{2} \right) \]

Thus, from (8), we have
\[ \phi \left( r_k \right) \leq \phi \left( \delta^*_{m(k)} + \delta^*_m \right) + 2\psi \left( r_k \right) - 4\psi \left( \frac{r_k}{2} \right) \]

Now using the properties of \( \phi \) and \( \psi \) and letting \( k \to +\infty \), we have
\[ \phi \left( r_k \right) \leq \phi \left( 0 \right) + 2\phi \left( \frac{r_k}{2} \right) - 4\lim_{k \to +\infty} \psi \left( \frac{r_k}{2} \right) \]

which is a contradiction. So, \( \{x_n\} \) and \( \{y_n\} \) are Cauchy sequences in the complete metric space \((X, p^*)\). Thus, by lemma 1 there are \( x, y \in X \) such that
\[ \lim_{n \to +\infty} p^* \left( x_n, x \right) = \lim_{n \to +\infty} p^* \left( y_n, y \right) = 0 \]

which implies that
\[ \lim_{n \to +\infty} F(x_n, y_n) = \lim_{n \to +\infty} x_n = x \]
\[ \lim_{n \to +\infty} F(y_n, x_n) = \lim_{n \to +\infty} y_n = y \]

Using (6), Lemma 1 and the property (p2), we have
\[ \lim_{n \to +\infty} p^* \left( x_n, x \right) = \lim_{n \to +\infty} p^* \left( y_n, y \right) = 0 \]

Suppose the condition (a) holds. Since \( F \) is continuous at \((x, y)\), for any \( \varepsilon > 0 \), \( \exists \delta > 0 \) such that if \( p(x, y) < \delta \) then \( F(x, y) = 0 \), so that
\[ p(F(x, y)) < p(F(x, y)) + p(x, y) - \delta = \delta \]

since \( p(x, y) = 0 \), then
\[ p(F(x, y), F(u, v)) < p(F(x, y), F(x, y)) + \frac{\delta}{2} \]

Since \( \lim_{n \to +\infty} p(x_n, x) = \lim_{n \to +\infty} p(y_n, y) = 0 \), for
\[ \eta = \min \left( \frac{\delta}{2}, \frac{\delta}{2} \right) > 0 \]

there exist \( n_0, m_0 \in N \) such that, for \( n \geq n_0, m \geq m_0 \), \( p(x_n, x) < \eta \) and \( p(y_m, y) < \eta \).

Then for \( n \in N, n \geq n_0 \), we have
\[ p(x_n, x) + p(y_m, y) < 2\eta < \delta \]
Theorem 2.
Suppose that there is a metric d on X. Where (X, ≤) is a partially ordered set such that (X, d) is a complete metric space. Suppose F : X → X is a nondecreasing mapping such that ∀ x, y ∈ X with y ≤ x, T(d(Fx, Fy), d(x, x), d(x, Fx), d(Fx, y), d(Fy, y)) ≤ 0, (13)

where T ∈ T. Also F is continuous, or if {xn} ⊂ X is a nondecreasing sequence with xn → x in X, then xn ≤ x ∀ n hold. If there exists an x0 ∈ X with x0 ≤ F (x0), then F has a fixed point.

Proof.
There is nothing to prove If Fx0 = x0, so suppose x0 # Fx0. Now let xn=Fxn-1 for n ∈ {1,2,...}. Notice that, since x0 ≤ Fx0 and F is nondecreasing, we have x0 ≤ x1 ≤ x2 ≤ ... ≤ xn ≤ x1 ≤ .......

Now since x0 ≤ x1 by inequality (13), we have T(d(Fx0,Fx1), d(x1, Fx1), d(x1, Fx1), d(x1, Fx1), d(x1, Fx1)) ≤ 0 so T(d(xn,n), d(xn,n), d(xn,n), d(xn,n), d(xn,n), d(xn,n), d(xn,n), d(xn,n)) ≤ 0.

By using T1, we have T(d(xn,n), d(xn,n), d(xn,n), d(xn,n), d(xn,n), d(xn,n), d(xn,n), d(xn,n)) ≤ 0. Using T2 a right continuous function f : x → x is exist, f(0) = 0, f(t) exist, for t>0 such that for all n ∈ {1,2,...}, d(xn,n) ≤ f(d(xn,n)).

If we continue this procedure, we can have d(xn+1,n) ≤ f(n) (dn+1,n), (14)

and so by lemma 2
\[ \lim_{n \to \infty} d (x_{n+1}, x_n) = 0. \]

Now this can be easily prove that {xn} is a Cauchy sequence in X, so ∃ an x ∈ X with limn→∞ x = x. then clearly x = Fx.

Suppose d(x, Fx)>0. Now since limn→∞ x = x, then xn ≤ x for all n. Using the inequality (13), we have T(d(Fx, Fx), d(x xn), d(x xn), d(x xn), d(x xn), d(x xn), d(x xn), d(x xn), d(x xn), d(x xn)) ≤ 0, so letting n → ∞ from the last inequality, we have T(d(Fx, x), 0, d (x, Fx), 0, d (x, Fx)) ≤ 0,

which is a contradiction to T3. Thus d (x, Fx) and so x = Fx.

Corollary
Suppose that there is a metric d on partially ordered set (X, ≤) and (X, d) is a complete metric space and F : X → X is a nondecreasing mapping such that ∀ x, y ∈ X with y ≤ x, d (Fx, Fy) ≤ max {d(x,y), d (x,Fx), d(y,Fy)} + (1-a) [ad (x, Fx)+bd (y,Fy)], where 0 ≤ a < 1, 0 ≤ a < 1/2, 0 ≤ b < 1/2.

Also F is continuous or if {xn} ⊂ X is a nondecreasing sequence with xn → x in X, then xn ≤ x ∀ n hold. If there exists an x0 ∈ X with x0 ≤ F (x0), then F has a fixed point.
Conclusion

In view of above results it is very much clear that we extend some fixed point results in partial metric space having the mixed monotone property and for nondecreasing mappings. This is the first effort in the existing literature.

Our results contain so many results in the existing literature and will be helpful for the workers in the field.

References


3. Shatanawi W., Samet B. and Abbas M., Coupled fixed point theorems for mixed monotone mappings in ordered partial metric spaces (2009)


