Research Journal of Mathematical and Statistical Sciences ________________________________ISSN 2320–6047
Vol. **1(8),** 25-27, September **(2013)** *Res. J. Mathematical & Statistical Sci.*

Characteristic and Moment Generating Functions of Three Parameter Weibull Distribution-an Independent Approach

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Available online at: www.isca.in
Received 30th July 2013, revised 22nd August 2013, accepted 2nd September 2013

Abstract

The characteristic function of three-parameter Weibull distribution is also derived independently and the moment generating function (MGF) is deduced from it. It generates all the moments of the distribution and satisfies the tests to verify a function to be a characteristic function. Expressions for mean, variance, skewness and kurtosis are also obtained from MGF.

Keywords: Three-parameter *Weibull* distribution, Characteristic function, Moment generating function.

Introduction

The characteristic function has many useful and important properties which give it a central role in statistical theory. It has great theoretical importance and also yields many valuable results in the theory of sampling[1]. The characteristic function is also known to be the inverse Fourier transforms of the probability density function.

Muraleedharan et.al[2] derived the characteristic function of two parameter *Weibull* distribution. Here another expression for the characteristic function (*CF*) of three-parameter *Weibull* distribution is derived independently and the moment generating function (*MGF*) is deduced from it. When the location parameter tends to zero, the *CF* of three parameter *Weibull* distribution tends to the *CF* of two parameter *Weibull* model form. Expressions for mean, variance, skewness and kurtosis are also obtained from *MGF*.

Mathematical Derivation of The Characteristic Function of Three Parameter *Weibull* Distribution

The characteristic function of three parameter *Weibull* distribution is given by $\varphi(t) = \mathrm{E}[\exp(itX)]$ (1)

E-usual expectation operator, *X*- *Weibull* random variable, $i = \sqrt{-1}$, *t*-any arbitrary real constant

The probability density function of three-parameter *Weibull* distribution is given by

$$f(x)\,dx = \frac{\lambda}{\sigma}\left(\frac{x-\mu}{\sigma}\right)^{\lambda-1} \exp\left[-\left(\frac{x-\mu}{\sigma}\right)^{\lambda}\right] dx,\ \mu < x < \infty \quad (2)$$

Where, μ, σ, λ >0, are respectively the location, scale and shape parameters of the distribution. Its cumulative distribution function is given by

$$F(x) = 1 - \exp\left[-\left(\frac{x-\mu}{\sigma}\right)^{\lambda}\right] \quad (3)$$

The characteristic function is mathematically derived as

$$\varphi(t) = \mathrm{E}[\exp(itX)] = \int_{\mu}^{\infty} \exp(itx) f(x)\,dx \quad (4)$$

$$= \underbrace{\int_{\mu}^{\infty} \cos(tx)\, f(x)\,dx}_{\mathrm{I}} + i \underbrace{\int_{\mu}^{\infty} \sin(tx)\, f(x)\,dx}_{\mathrm{II}}$$

Integrating the **I**st expression by parts, $\Rightarrow$

$$\int_{\mu}^{\infty} \cos(tx) f(x)\, dx = \cos(t\mu) - \int_{\mu}^{\infty} t \sin(tx) \exp\left[-\left(\frac{x-\mu}{\sigma}\right)^{\lambda}\right] dx \qquad \text{A}$$

Now integrating the **II**nd expression by parts $\Rightarrow$

$$i\int_{\mu}^{\infty} \sin(tx) f(x)\, dx = i\sin(t\mu) + \int_{\mu}^{\infty} it \cos(tx) \exp\left[-\left(\frac{x-\mu}{\sigma}\right)^{\lambda}\right] dx \qquad \text{B}$$

A + B $\Rightarrow$, $\int_{\mu}^{\infty}[\cos(tx) + i\sin(tx)] f(x)\, dx = \exp(it\mu) + \sum_{r=0}^{\infty} \frac{(it)^{r+1}}{r!} \int_{\mu}^{\infty} x^{r} \exp\left[-\left(\frac{x-\mu}{\sigma}\right)^{\lambda}\right] dx$

Putting $\left(\frac{x-\mu}{\sigma}\right)^{\lambda} = y$ and integrating leads to the characteristic function of three parameter *Weibull* distribution, $\varphi(t)$:

$$\varphi(t) = \exp(it\mu) + \frac{\sigma}{\lambda}\sum_{r=0}^{\infty} \frac{(it)^{r+1}}{r!} \mu^{r} \sum_{n=0}^{r} {}^{r}C_{n} \left(\frac{\sigma}{\mu}\right)^{n} \Gamma\left(\frac{n+1}{\lambda}\right), r = 0, 1, 2, ... \qquad (5)$$

Where

$$ {}^{r}C_{n} = \frac{r!}{n!(r-n)!} \qquad (6)$$

The derivation is complete and the expression is new. The methodology is direct and elegant. Its expansion will be of the form

$$\varphi(t) = \left(1 + \frac{it\mu}{1!} - \frac{t^{2}\mu^{2}}{2!} - ...\right) + \frac{\sigma}{\lambda}\left\{\begin{array}{l} it\left[\Gamma\left(\frac{1}{\lambda}\right)\right] - t^{2}\mu\left[\Gamma\left(\frac{1}{\lambda}\right) + \left(\frac{\sigma}{\mu}\right)\Gamma\left(\frac{2}{\lambda}\right)\right] - \\ \frac{it^{3}\mu^{2}}{2}\left[\Gamma\left(\frac{1}{\lambda}\right) + 2\left(\frac{\sigma}{\mu}\right)\Gamma\left(\frac{2}{\lambda}\right) + \left(\frac{\sigma}{\mu}\right)^{2}\Gamma\left(\frac{3}{\lambda}\right)\right] + \frac{t^{4}\mu^{3}}{3!}[...] + ... \end{array}\right\} \qquad (7)$$

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It also satisfies the tests to verify whether a function is a characteristic function[3, 4]. Ie., i. That $\varphi(t)$ must be continuous in *t,* ii. That $\varphi(t)$ is defined in every finite *t* interval, iii. That $\varphi(0) = 1$, iv. That $\varphi(t)$ and $\varphi(-t)$ shall be conjugate quantities.

When μ (location parameter)$\rightarrow 0$, then

$$\varphi(t) = 1 + it\sigma\,\Gamma\left(1 + \frac{1}{\lambda}\right) - \frac{t^{2}\sigma^{2}}{2!}\Gamma\left(1 + \frac{2}{\lambda}\right) - \frac{it^{3}\sigma^{3}}{3!}\Gamma\left(1 + \frac{3}{\lambda}\right) + ... \qquad (8)$$

Or

$$\varphi(t) = \sum_{r=0}^{\infty} \frac{(it\sigma)^{r}}{r!}\Gamma\left(1 + \frac{r}{\lambda}\right) \qquad (9)$$

Expression 9 is the characteristic function of two parameter *Weibull* distribution[1, 2]. Ie. When $\mu \rightarrow 0$, *CF* of three parameter *Weibull* distribution tends to *CF* of the two parameter model.

Discussion

In many cases, the moment generating functions (*MGF*) can be easily deduced from characteristic functions. Ie. by arranging $it=\theta$,

$$M(\theta) = \exp(\theta\mu) + \frac{\sigma}{\lambda}\sum_{r=0}^{\infty} \frac{(\theta)^{r+1}}{r!} \mu^{r} \sum_{n=0}^{r} {}^{r}C_{n} \left(\frac{\sigma}{\mu}\right)^{n} \Gamma\left(\frac{n+1}{\lambda}\right) \qquad (10)$$

which is the moment generating function of the three parameter *Weibull* distribution. *M (θ)* can generate all the raw moments of the distribution. Ie

$$M_X^{(n)}(0) = \mu_n' = E\left(X^n\right) = \left[\left(\frac{d^n\left(M(\theta)\right)}{d\theta^n}\right)\right]_{\theta=0} \tag{11}$$

μ_n' - n^{th} arbitrary moment or n^{th} raw moment. Accordingly

$$Mean = \mu_1' = \mu + \sigma\Gamma\left(1+\frac{1}{\lambda}\right) \tag{12}$$

$$Var\,(x) = \mu_2 = \mu_2' - \left(\mu_1'\right)^2 = \sigma^2\left[\Gamma\left(1+\frac{2}{\lambda}\right) - \Gamma^2\left(1+\frac{1}{\lambda}\right)\right] \tag{13}$$

$$Skewness = \frac{\mu_3}{\left(\mu_2\right)^{\frac{3}{2}}} = \frac{g_3 - 3g_1g_2 + 2g_1^3}{\left(g_2 - g_1^2\right)^{\frac{3}{2}}} \tag{14}$$

$$Kurtosis = \frac{\mu_4}{\left(\mu_2\right)^2} = \frac{g_4 - 4g_1g_3 + 6g_2g_1^2 - 3g_1^4}{\left(g_2 - g_1^2\right)^2} \tag{15}$$

μ_2, μ_3 *and* μ_4 are the central moments. The variance of two and three parameter *Weibull* distributions are the same.

$$\mu_3' = \mu^3 + 3\mu^2\sigma\, g_1 + 3\mu\sigma^2 g_2 + \sigma^3 g_3 \tag{16}$$

$$\mu_4' = \mu^4 + 4\mu^3\sigma\, g_1 + 6\mu^2\sigma^2 g_2 + 4\mu\sigma^3 g_3 + \sigma^4 g_4 \tag{17}$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2\left(\mu_1'\right)^3 \tag{18}$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\left(\mu_1'\right)^2 - 3\left(\mu_1'\right)^4 \tag{19}$$

$$g_k = \Gamma\left(1+\frac{k}{\lambda}\right) \tag{20}$$

Conclusion

The three parameter *Weibull* distribution is widely used in extreme event modeling. Hence it's *CF* is derived and it satisfies all the conditions for a function to be a characteristic function. It is able to generate all the moments of 3-parameter *Weibull* distribution from its *MGF*. Hence expressions for mean, variance, skewness and kurtosis are also obtained from *MGF*.

Acknowledgement

The work of the author has been funded by The University of Lisbon, Instituto Superior Tecnico, Lisboa, Portugal.

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