Fixed Point Results on Fuzzy Mappings for Rational Expressions

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Abstract

In this paper we prove some fixed point and common fixed point theorems for fuzzy mappings in complete metric space which also include rational expression as a contraction. AMS Subject Classification: 54H25, 47H10

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Introduction

In 1965 Zadeh\(^1\) introduce the concept of fuzzy sets. After that so many works have been done in fuzzy sets. In 1981 Heilpern\(^2\) define fuzzy mappings are use of fuzzy mappings he proved fixed point theorem which is a fuzzy analogue of the fixed point theorem for multi valued mappings of Nadler\(^3\), Vijayaraju and Marudai\(^4\) generalized the results of Bose and Mukherjee\(^5\) for fuzzy mappings. So many authors Marudai and Srinivasan\(^6\), Bose and Sahani\(^7\), Butnariu\(^8,9,10\), Chang and Huang\(^11\), Chang\(^12\), Chitra\(^13\), Som and Mukharjee\(^14\) studied fixed point theorems for fuzzy mappings. Lee, Cho, Lee and Kim\(^15\) obtained a common fixed point theorem for a sequence of fuzzy mappings satisfying certain conditions, which is generalization of the second theorem of Bose and Sahini\(^7\).

More recently Vijayaraju and Mohanraj\(^16\) obtained some fixed point theorems for contractive type fuzzy mappings which are generalization of Beg and Azam\(^17\). Fuzzy extension of Kirk and Downing\(^18\) and its prove analogue to the proof of Park and Jeong\(^19\). In the present paper we are proving some fixed point and common fixed point theorems in fuzzy mappings containing the rational expressions.

Preliminaries

We need following definitions and assumptions:

**Definition 2.1** Let X be any metric linear space and d be any metric on X. A fuzzy set in X is a function with domain X and values in [0,1]. If A is a fuzzy set and \(x\in X\), the function value A(x) is called the grade of membership of x in A. The collection of all fuzzy sets in X is denoted by \(\mathcal{F}(X)\).

Let \(A\in \mathcal{F}(X)\) and \(\alpha\in [0,1]\). The set \(\alpha\)-level set of A, denoted by \(A_\alpha\)

\[A_\alpha = \{ x: A(x) \geq \alpha \} \text{ if } \alpha \in [0,1],\]

\[A_\alpha = \{ x: A(x) > 0 \} \text{ whenever } B \text{ is clouser of } B\]

Now we distinguish from the collection \(\mathcal{F}(X)\) a sub collection of approximate quantities, denoted \(W(X)\).

**Definition 2.2** A fuzzy subset A of X is an approximate quantity if its \(\alpha\)-level set is a compact subset (non fuzzy) of X for each \(\alpha \in [0,1]\), and \(\sup_{x\in X} A(x) = 1\).

Let \(A \in W(X)\) and \(A(x_0) = 1\) for some \(x_0 \in W(X)\), we will identify A with an approximation of \(x_0\). Then we shall define a distance between two approximate quantities.

**Definition 2.3** Let A, B \(\in W(X)\), \(\alpha \in [0,1]\), define

\[p_\alpha (A, B) = \inf_{x\in A_\alpha, y\in B_\alpha} d(x, y), D_\alpha (A, B) = \text{dist}(A_\alpha, B_\alpha), d(A, B) = \sup_\alpha D_\alpha (A, B)\]

Where dist. Is Hausdorff distance. The function \(p_\alpha\) is called \(\alpha\)-spaces, and a distance between A and B. It is easy to see that \(p_\alpha\) is non decreasing function of \(\alpha\). We shall also define an order of the family \(W(X)\), which characterizes accuracy of a given quantity.
Definition 2.4 Let A, B ∈ W(x). An approximate quantity A is more accurate than B, denoted by A ⊆ B, iff A(x) ≤ B(x), for each x ∈ X.

Definition 2.5 Let X be an arbitrary set and Y be any metric linear space. F is called a fuzzy mapping if F is mapping from the set X into W(Y), i.e., F(x) ∈ W(Y) for each x ∈ X.

A fuzzy mapping F is a fuzzy subset on X x Y with membership function F(x,y). The function value F(x,y) is grade of membership of y in F(x).

Let A ∈ F(X), B ∈ F(Y), the fuzzy set $F^{-1}(B)$ in F(X), is defined as $F^{-1}(B) = \{x ∈ X : F(x) ∈ B\}$.

First of all we shall give here the basic properties of α-space and α-distance between some approximate quantities.

Lemma 2.1: Let x ∈ X, A ∈ W(X), and \{x\} be a fuzzy set with membership function equal a characteristic function of set \{x\}. If \{x\} is subset of A then $p_\alpha(x,A) = 0$ for each $\alpha ∈ [0,1]$.

Lemma 2.2: $p_\alpha(x,A) ≤ d(x,y) + p_\alpha(y,A)$ for any x, y ∈ X.

Lemma 2.3: If \{x_0\} is subset of A, then $p_\alpha(x_0,B) ≤ D_\alpha(A,B)$ for each B ∈ W(X).

Lemma 2.4: Let (X,d) be a complete metric space, T be a fuzzy mapping from X into W(X), then there exists $x_1 ∈ X$ such that \{x_1\} ⊂ T{\{x_0\}}.

Lemma 2.5: Let A, B ∈ W(X). Then for each \{x\} ⊂ A, there exists \{y\} ⊂ B such that $D(\{x\}, \{y\}) ≤ D(A,B)$.

Definition (2.6): An intuitionist fuzzy set (i-fuzzy set) $A$ of X is an object having the form $A = (A_1, A_2)$, where $A_1, A_2 ∈ [0,1]$ and $A_1(x) + A_2(x) ≤ 1$ for each $x ∈ X$. We denote by IFS(X) the family of all i-fuzzy sets of X.

Definition (2.7): Let $x_α$ be a fuzzy point of X. We will say that $x_α$ is a fixed fuzzy point of the fuzzy mapping $F$ over X if $x_α ⊆ F(x)$. In particular and according to $2$, if \{x\} ⊂ F(x), we say that x is a fixed point of $F$.

Remark 2.1: Notice $[x_\alpha] ⊂ A$ if and only if $x_0 ⊂ A_1$.
Main Results

**Theorem 3.1:** Let $(X, d)$ be a complete metric space. Let $F$ be continuous fuzzy mapping from $X$ into $W_\alpha(X)$ satisfying the following condition: There exists $K \in (0,1]$ such that

$$d(x, y) \leq K \left( d(x, y) + p_\alpha(x, F(x)) \right)$$

for all $x, y \in X$ with $x \neq y$, and

$$d(x, y) \leq K \left( d(x, y) + p_\alpha(x, F(y)) \right)$$

Then there exists $x \in X$ such that $x_0$ is a fixed fuzzy point of $F$ if and only if $x_0 \in G(x)$ with

$$\sum_{n=1}^{\infty} K^n d(x_0, x_1) < \infty$$

for $\alpha \in (0,1]$. In particular if $\alpha = 1$ then $x$ is a fixed point of $F$.

**Proof:** If there exists $x \in X$ such that $x_0 \in G(x)$, i.e. $x_0 \subset G(x)$, then $\sum_{n=1}^{\infty} K^n d(x_0, x_1) = 0$. Let $x_0 \in K$ and suppose that there exists $x_1 \in (F(x_0))_\alpha$ such that

$$d(x_1, x_2) = p_\alpha(x_1, F(x_1)) \leq D_\alpha(F(x_0), F(x_1))$$

By induction we construct a sequence $\{x_n\}$ in $X$ such that $x_n \subset (F(x_{n+1}))_\alpha$, and $d(x_n, x_{n+1}) \leq d(x_0, x_1)$. Since $K$ is given to be the non-decreasing, so

$$d(x_{n+1}, x_{n+2}) \leq K \{d(x, y)\}$$

Therefore the sequence $\{x_n\}$ is a Cauchy sequence in $X$ and $X$ is complete therefore $\{x_n\}$ converges to $x \in X$. By the help of lemma 2.1 and 2.2 we have
\[ p_\alpha(x, F(x)) \leq d(x, x_n) + p_\alpha(x_n, F(x)) \leq d(x, x_n) + D_\alpha(F(x_{n-1}), F(x)) \leq d(x, x_n) + Kd(x_{n-1}, x) \]  
 Consequently, \( p_\alpha(x, F(x)) = 0 \), and by lemma 2.1 \( x_\alpha \subseteq F(x) \) Clearly \( x_\alpha \) is a fixed fuzzy point of the fuzzy mapping \( F \) over \( X \). In particular if \( \alpha = 1 \) then \( x \) is a fixed point of \( F \). Now we will generalize this theorem for common fixed point.

**Theorem 3.2:** Let \((X, d)\) be a complete metric space. Let \( T \) and \( S \) be continuous fuzzy mappings from \( X \) into \( W_\alpha(X) \) and \( F : X \to W_\alpha(X) \) be a mapping such that 
\[ \{ M(x, y) \mid K \phi \left[ D_a(Sx, Ty), D_a(Sx, Fx), D_a(Ty, Fy), D_a(Fx, Ty) \right] \leq M(x, y) \leq \frac{D_a(Sx, Ty) + D_a(Fx, Ty)}{1 + D_a(Sx, Ty) - D_a(Fx, Ty)} \] 
Where \( K \) is non-decreasing function such that \( K : [0, \infty) \to [0, \infty) \).

\( K(0) = 0 \) and \( K(t) < t \) \( \forall t \in (0, \infty) \), \( \alpha \in (0, 1] \) and then \( \exists x \in X \) such that \( x_\alpha \) is a common fixed fuzzy point of \( S, T \) and \( F \) if and only if \( x_0, x_1 \in X \) such that 
\[ \sum_{n=1}^{\infty} K^n d(x_0, x_1) < \infty. \]
In particular if \( \alpha = 1 \), then \( x \) is a common fixed point of \( S, T \) and \( F \).

**Proof:** Let for \( x_0 \in X \) there exists \( x_1 \) and \( x_2 \) such that \( x_1 \in (S(x_1))_\alpha \subset (F(x_0))_\alpha \) and \( x_2 \in (T(x_2))_\alpha \subset (F(x_1))_\alpha \). By induction one can construct a sequence \( \{x_n\} \) in \( X \) such that 
\[ x_{2n+1} \in \left(Sx_{2n+1}\right)_\alpha \subset \left(F(x_{2n+1}\right)_\alpha . \text{And} \ x_{2n+2} \in \left(Tx_{2n+2}\right)_\alpha \subset \left(F(x_{2n+2}\right)_\alpha . \]

Since \( K \) is given to be non-decreasing. So 
\[ d(x_n, x_{n+1}) \leq D_\alpha(F(x_{n-1}), F(x_n)) \leq K\times M(x_{n-1}, x_n) \]
\[ = K \phi \left[ D_a(Sx_{n-1}, Tx_n), D_a(Sx_{n-1}, Fx_n), D_a(Tx_n, Fx_n), D_a(Fx_{n-1}, Tx_n), \right] \]
\[ = K \phi \left[ d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_{n-1}, x_n), d(x_{n-1}, x_n) \right] \]
\[ = K \{d(x_{n-1}, x_n)\} \]
Therefore 
\[ d(x_{n-1}, x_{n+1}) \leq Kd(x_{n-1}, x_n) = Kd_\alpha(F(x_{n-2}), F(x_{n-1})) \leq K^2d(x_{n-2}, x_{n-2}) \leq \ldots \leq K^nd(x_0, x_1) \]
\[ \Rightarrow d(x_n, x_{n+m}) \leq d(x_n, x_{n+1}) + \ldots + d(x_{n+m-1}, x_{n+m}) \leq K^md(x_0, x_1) + \ldots + K^{n-1}d(x_0, x_1) = \sum_{j=0}^{m-1} K^j d(x_0, x_1) \]
Since \( \sum_{n=1}^{\infty} K^n d(x_0, x_1) \leq \infty \) it follows that there exists \( u \) such that \( d(x_n, x_{n+m}) < u \in X \). Therefore the sequence \( \{x_n\} \) is a Cauchy sequence in \( X \). Since \( X \) is complete, \( \{x_n\} \) converges to \( x \in X \) and \( \left(Sx_{2n+1}\right)_\alpha , \left(Tx_{2n+2}\right)_\alpha \) also converges on \( X \).
Since \( \{S, F\} \) and \( \{T, F\} \) are weakly commuting mappings, so \( p_\alpha(x, F(x)) \leq d(x, x_n) + p_\alpha(x_n, F(x)) \leq d(x, x_n) + D_\alpha(F(x_{n-1}), F(x)) \leq d(x, x_0) + Kd(x_{n-1}, x) \). Consequently, \( p_\alpha(x, F(x)) = 0 \) and by lemma 2.1, \( x_\alpha \subseteq F(x) \).

**Conclusion**

Clearly \( x_\alpha \) is a common fixed fuzzy point of the fuzzy mapping \( F, S \) and \( T \) over \( X \). In particular if \( \alpha = 1 \) then \( x \) is a common fixed point of \( F, S \) and \( T \).

**References**